The Pricing Mechanism of Contingent Claims and its Generating Function

Shige PENG*
Institute of Mathematics, Institute of Finance
Shandong University
250100, Jinan, China
peng@sdu.edu.cn

Version: March 31, 2006

Contents

*The author thanks the partial support from the Natural Science Foundation of China, grant No. 10131040. This research is supported in part by The National Natural Science Foundation of China No. 10131040. This reversion is made after the author's visit, during November 2003, to Institute of Mathematics and System Science, Academica Sinica, where he gives a series of lectures on this paper. He thanks Zhiming Ma and Jia-an Yan, for their fruitful suggestions, critics and warm encouragements. He also thanks to Claude Dellacherie for his suggestions and critics.

Abstract. In this paper we study dynamic pricing mechanism of contingent claims. A typical model of such pricing mechanism is the so-called g–expectation $E^g_{s,t}[X]$ defined by the solution of the backward stochastic differential equation with generator g and with the contingent claim X as terminal condition. The generating function g this BSDE. We also provide examples of determining the price generating function $g = g(y, z)$ by testing.

The main result of this paper is as follows: if a given dynamic pricing mechanism is $\mathbb{E}^{g_\mu}$–dominated, i.e., the criteria (A5) $\mathbb{E}_{s,t}[X] - \mathbb{E}_{s,t}[X'] \leq \mathbb{E}^{g_\mu}[X - X']$ is satisfied for a large enough $\mu > 0$, where $g_\mu = \mu(|y| + |z|)$, then $\mathbb{E}_{s.t}[\cdot]$ is a g–pricing mechanism. This domination condition was statistically tested using CME data docoments. The result of test is significantly positive.

Keywords: BSDE, nonlinear expectation, dynamic pricing mechanism, g–expectation, nonlinear evaluation, g-martingale, nonlinear martingale, Doob-Meyer decomposition.

MSC 2000 Classification Numbers: 60H10, 60H05,

1 Introduction

There are a lot of data of the processes of prices of huge variety of contingent claims, vanilla options, exotic options, etc. Each process corresponds the price of a specific contingent claim issued in a specific market and offered by a specific financial institution. A typical example is the call and put options with a specific stock price as their underlying asset. We can find the real time data of of the option price C_t, $t \geq t_0$ for a call option $C_T = (S_T - k)^+$ with T as its maturity. There exist many processes of prices of this specific product, e.g., the bid price, the ask price, the we-buy price and we-sell price by a market maker under a specific background, etc. The main point of view of this paper is, behind a price process, there is a pricing mechanism. Take the above option market price C_t for example, there exists a mapping $\mathbb{E}_{t,T}[\cdot]$ from Λ_T the space of option price states at time T to Λ_t at the time $t \in [t_0, T]$ such that C_t is produced by $\mathbb{E}_{t,T}[C_T]$. This family of mapping

$$\mathbb{E}_{t,T}[X] : X \in \Lambda_T \longmapsto \Lambda_t, \ t \leq T$$

forms the pricing mechanism for this specific option market prices.

Black-Scholes formula can be regarded as a dynamic pricing mechanism of contingent claim. In fact, it can be regarded as to solve a specific linear backward stochastic differential equation (BSDE). More generally, each BSDE with a given generating function g forms a dynamic model of pricing mechanism of contingent claims.

In this paper we explain the following result: if an a dynamic pricing mechanism is dominated by g_μ–pricing mechanism, with large enough $\mu > 0$, then it is a g–pricing mechanism: there exists a unique generating function g, such that the price of the pricing mechanism is solved by the corresponding BSDE. In this case, to find the corresponding generating function g by using data of the pricing process is a very interesting problem, since g determines entirely the pricing mechanism. The domination condition can be tested also by data analysis of the price processes.

The paper is organized as follows: in section 2, we present the the notion of $\mathcal{F}_t$–consistent pricing mechanisms in subsection 2.1. We then give a concrete $\mathcal{F}_t$–pricing mechanism: $\mathbb{E}^g$–pricing mechanisms in subsection 2.2. The main result, Theorem 3.1, will be presented in section 3. We also provide some examples and explain how to find the function g through by testing the input–output data. This main theorem will be proved in Section 9. Nonlinear decomposition theorems of Doob–Meyer's type, i.e., Proposition 4.13 and Theorem 8.1 play crucial roles in the proof of Theorem 3.1. Theorem 8.1 has also an interesting interpretation in finance (see Remark 8.2).

The crucial domination inequality (3.1) of our main result Theorem 3.1 is tested by using data of parameter files, provided by CME, for options based on S&P500 futures. The result strongly support that the option pricing mechanism of CME is a g–pricing mechanism.

Another application of the dynamical expectations and pricing mechanisms is to risk measures. Axiomatic conditions for a (one step) coherent risk measure was introduced by Artzner, Delbaen, Eber and Heath 1999 [2] and, for a convex risk measure, by Föllmer and Schied (2002) [26]. Rosazza Gianin (2003) studied dynamical risk measures using the notion of g–expectations in [45] (see also [41], [3], [4]) in which (B1)–(B4) are satisfied. In fact conditions (A1)-(A4), as well as their special situation (B1)–(B4) provides an ideal characterization of the dynamical behaviors of a the a risk measure. But in this paper we emphasis the study of the mechanism of the pricing mechanism to a further payoff, for which is, in general, the translation property in risk measure is not satisfied.

2 The pricing mechanisms and g–pricing mechanism by BSDE

2.1 Basic setting

We assume that the price S of the underlying assets is driven by a d–dimensional Brownian motion $(B_t)_{t\geq 0}$ in a probability space $(\Omega, \mathcal{F}, P)$. We don't need to

precise the model of S_t, what we assume here is that the information $\mathcal{F}_t^S$ of the price S coincides with that of the Brownian motion:

$$\mathcal{F}_t^S = \mathcal{F}_t := \sigma\{B_s,\ s \le t\}$$

For each $t \in [0, \infty)$, the state of contingent prices will be given in the following space

- $\Lambda_t = L^2(\mathcal{F}_t)$:={the space of all real valued $\mathcal{F}_t$–measurable random variables such that $E[|X|^p] < \infty$}.

Definition 2.1 *A system of operators:*

$$\mathbb{E}_{s,t}[X] : X \in L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s),\ T_0 \le s \le t \le T_1$$

is called an $\mathcal{F}_t$–consistent pricing mechanism defined on $[T_0, T_1]$ if it satisfies the following properties: for each $T_0 \le s \le t \le T_1$ and for each X_t, $X_t' \in L^2(\mathcal{F}_t)$,
(A1) $\mathbb{E}_{s,t}[X_t] \ge \mathbb{E}_{s,t}[X_t']$*, a.s., if* $X_t \ge X_t'$*, a.s.;*
(A2) $\mathbb{E}_{t,t}[X_t] = X_t$*, a.s.;*
(A3) $\mathbb{E}_{r,s}[\mathbb{E}_{s,t}[X_t]] = \mathbb{E}_{r,t}[X_t]$*, a.s.; for* $r \le s$
(A4) $1_A\mathbb{E}_{s,t}[X_t] = 1_A\mathbb{E}_{s,t}[1_A X_t]$*, a.s.* $\forall A \in \mathcal{F}_s$*.*

We will often consider (A1)–(A4) plus an additional condition:
(A4$_0$) $\mathbb{E}_{s,t}[0] = 0$, a.s. $\forall 0 \le s \le t \le T$.

Remark 2.2 *The raison we use the letter $\mathbb{E}_{s,t}[\cdot]$ to denote the above pricing mechanism is that its behavior is very like the conditional expectation $E[X_t|\mathcal{F}_s]$ for a $\mathcal{F}_t$–measurable random variable. It is wise profit this similarity to introduce the notion of $\mathbb{E}$-martingales which are the data of the processes of option prices produced by this pricing mechanism.*

Remark 2.3 *(A1) and (A2) are economically obvious conditions. Condition (A3) means that the value $\mathbb{E}_{s,t}[X_t]$ can be regarded as a contingent claim at the maturity s. The price of this contingent claim at the time $r \le s$ is $\mathbb{E}_{r,s}[\mathbb{E}_{s,t}[X_t]]$. It have to be the same as the price $\mathbb{E}_{r,t}[X_t]$.*

Remark 2.4 *The meaning of condition (A4) is: at time s, the agent knows whether I_A worthes 1. If it is 1, then the price $\mathbb{E}_{s,t}[1_A X_t]$ must be the same as $\mathbb{E}_{s,t}[X_t]$.*

Proposition 2.5 *(A4) plus (A4$_0$) is equivalent to*
(A4') $1_A\mathbb{E}_{s,t}[X] = \mathbb{E}_{s,t}[1_A X]$*, a.s.* $\forall A \in \mathcal{F}_s$*.*

Proof. . It is clear that (A4') implies (A4). $\mathbb{E}_{s,t}[0] \equiv 0$ can be derived by putting $A = \emptyset$ in (A4'). On the other hand, (A4) plus the additional condition implies

$$1_{A^C}\mathbb{E}_{s,t}[1_A X] = 1_{A^C}\mathbb{E}_{s,t}[1_{A^c}1_A X] = 0.$$

We thus have

$$\begin{aligned}\mathbb{E}_{s,t}[1_A X] &= 1_{A^C} 1_A \mathbb{E}_{s,t}[X] + 1_A 1_A \mathbb{E}_{s,t}[X] \\ &= 1_A \mathbb{E}_{s,t}[X].\end{aligned}$$

$\blacksquare$

Proposition 2.6 *(A4) is equivalent to, for each* $0 \le s \le t$ *and* $X, X' \in L^2(\mathcal{F}_t)$*,*

$$\mathbb{E}_{s,t}[1_A X + 1_{A^C} X'] = 1_A \mathbb{E}_{s,t}[X] + 1_{A^C} \mathbb{E}_{s,t}[X'], \text{ a.s. } \forall A \in \mathcal{F}_s. \tag{2.1}$$

Proof. (A4) $\Rightarrow$ (2.1): We let $Y = 1_A X + 1_{A^C} X'$. Then, by (A4)

$$1_A \mathbb{E}_{s,t}[Y] = 1_A \mathbb{E}_{s,t}[1_A Y] = 1_A \mathbb{E}_{s,t}[1_A X] = 1_A \mathbb{E}_{s,t}[X].$$

Similarly

$$1_{A^C} \mathbb{E}_{s,t}[Y] = 1_{A^C} \mathbb{E}_{s,t}[1_{A^C} Y] = 1_{A^C} \mathbb{E}_{s,t}[1_{A^C} X'] = 1_{A^C} \mathbb{E}_{s,t}[X'].$$

Thus (2.1) from $1_A \mathbb{E}_{s,t}[Y] + 1_{A^C} \mathbb{E}_{s,t}[Y] = 1_A \mathbb{E}_{s,t}[X] + 1_{A^C} \mathbb{E}_{s,t}[X']$. $\blacksquare$

(2.1) $\Rightarrow$ (A4): It is simply because of

$$\begin{aligned}1_A \mathbb{E}_{s,t}[1_A X] &= 1_A \mathbb{E}_{s,t}[1_A X + 1_{A^C}(1_A X)] \\ &= 1_A (1_A \mathbb{E}_{s,t}[X] + 1_{A^C} \mathbb{E}_{s,t}[1_{A^C} X]) \\ &= 1_A \mathbb{E}_{s,t}[X].\end{aligned}$$

Remark 2.7 *At time* t*, the agent knows the value of* 1_A*. (A4) means that, if,* $\omega \in A$*, i.e.., * $1_A(\omega) = 1$ *then the value* $\mathbb{E}_{s,t}[1_A X]$ *should be the same as* $\mathbb{E}_{s,t}[X]$ *since the two outcomes* $X(\omega)$ *and* $(1_A X)(\omega)$ *are exactly the same. (A4) is applied to the pricing mechanism of a final outcome* X *plus some "dividend"* $(D_s)_{s \ge 0}$*.*

An immediate property of this dynamical pricing mechanism is that they can be pasted together, one after the other to form a new dynamical pricing mechanism:

Proposition 2.8 *Let* $T_0 < T_1 < T_2 < \cdots < T_N$ *be given and, for* $i = 0, 1, 2, \cdots, N-1$*, let* $\mathbb{E}^i_{s,t}[\cdot]$*,* $T_i \le s \le t \le T_{i+1}$ *be an* $\mathcal{F}_t$*–consistent pricing mechanism on* $[T_i, T_{i+1}]$ *in the sense of Definition 2.1. Then there exists a unique* $\mathcal{F}_t$*–consistent pricing mechanism* $\mathbb{E}[\cdot]$ *defined on* $[T_0, T_N]$

$$\mathbb{E}_{s,t}[X] : X \in L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s),\ T_0 \le s \le t \le T_N$$

such that, for each $i = 0, 1, \cdots, N-1$*, and for each* $T_i \le s \le t \le T_{i+1}$*,*

$$\mathbb{E}_{s,t}[X] = \mathbb{E}^i_{s,t}[X],\ \forall X \in L^2(\mathcal{F}_t). \tag{2.2}$$

Proof. It suffices to prove the case $N = 2$, since we then can apply this result to the cases $[T_0, T_3] = [T_0, T_2] \cup [T_2, T_3]$, $\cdots$ and finally $[T_0, T_N] = [T_0, T_{N-1}] \cup [T_{N-1}, T_N]$.
We define

$$\mathbb{E}_{s,t}[X] = \begin{cases} \mathbb{E}^1_{s,t}[X] & T_0 \le s \le t \le T_1; \\ \mathbb{E}^2_{s,t}[X], & T_1 \le s \le t \le T_2; \\ \mathbb{E}^1_{s,T_1}[\mathbb{E}^2_{T_1,t}[X]] & T_1 \le s < T_1 < t \le T_2. \end{cases} \tag{2.3}$$

It is clear that, on $[T_0, T_2]$, $\mathbb{E}_{s,t}[\cdot]$ satisfies (A1) and (A2). To prove (A3) it suffices to check the relation

$$\mathbb{E}_{r,s}[\mathbb{E}_{s,t}[X]] = \mathbb{E}_{r,t}[X],\ T_0 \le r \le s \le t \le T_1$$

for two cases: $T_0 \le r \le s \le T_1 \le t \le T_2$ and $T_0 \le r \le T_1 \le s \le t \le T_2$. For the first case

$$\begin{aligned} \mathbb{E}_{r,s}[\mathbb{E}_{s,t}[X]] &= \mathbb{E}^1_{r,s}[\mathbb{E}^1_{s,T_1}[\mathbb{E}^2_{T_1,t}[X]]] \\ &= \mathbb{E}^1_{r,T_1}[\mathbb{E}^2_{T_1,t}[X]] \\ &= \mathbb{E}_{r,t}[X]. \end{aligned}$$

For the second case

$$\begin{aligned} \mathbb{E}_{r,s}[\mathbb{E}_{s,t}[X]] &= \mathbb{E}^1_{r,T_1}[\mathbb{E}^2_{T_1,s}[\mathbb{E}^2_{s,t}[X]]] \\ &= \mathbb{E}^1_{r,T_1}[\mathbb{E}^2_{T_1,t}[X]] \\ &= \mathbb{E}_{r,t}[X]. \end{aligned}$$

We now prove (A4). Again it suffices to check the case $T_0 \le s \le T_1 \le t \le T_2$. In this case, for each $A \in \mathcal{F}_s \subset \mathcal{F}_{T_1}$, (A4) is derived from

$$\begin{aligned} 1_A \mathbb{E}_{s,t}[X] &= 1_A \mathbb{E}^1_{s,T_1}[\mathbb{E}^2_{T_1,t}[X]] \\ &= 1_A \mathbb{E}^1_{s,T_1}[1_A \mathbb{E}^2_{T_1,t}[X]] \\ &= 1_A \mathbb{E}^1_{s,T_1}[\mathbb{E}^2_{T_1,t}[1_A X]] \\ &= 1_A \mathbb{E}_{s,t}[1_A X]. \end{aligned}$$

It remains to prove the uniqueness of $\mathbb{E}[\cdot]$. Let $\mathbb{E}^a[\cdot]$ be an $\mathcal{F}_t$–consistent pricing mechanism such that,

$$\mathbb{E}^a_{s,t}[X] = \mathbb{E}^i_{s,t}[X],\ \forall X \in L^2(\mathcal{F}_t),\ i = 1, 2.$$

We then have, when $T_0 \le s \le t \le T_1$ and $T_1 \le s \le t \le T_2$, $\mathbb{E}^a_{s,t}[X] \equiv \mathbb{E}_{s,t}[X]$, $\forall X \in L^2(\mathcal{F}_t)$. For the remaining case, i.e., $T_0 \le s < T_1 < t \le T_1$, since $\mathbb{E}^a$ satisfies (A3),

$$\begin{aligned} \mathbb{E}^a_{s,t}[X] &= \mathbb{E}^a_{s,T_1}[\mathbb{E}^a_{T_1,t}[X]] \\ &= \mathbb{E}^1_{s,T_1}[\mathbb{E}^2_{T_1,t}[X]] \\ &= \mathbb{E}_{s,t}[X],\ \forall X \in L^2(\mathcal{F}_t). \end{aligned}$$

Thus $\mathbb{E}^a_{s,t}[\cdot] = \mathbb{E}_{s,t}[\cdot]$. This completes the proof. ■

2.2 Dynamic pricing mechanism generated by BSDE

We need the following notations. Let $p \geq 1$ and $t \in [0, \infty)$ be given.

- $L^p(\mathcal{F}_t; R^m)$:={the space of all R^m–valued $\mathcal{F}_t$–measurable random variables such that $E[|\xi|^p] < \infty$};
- $L^p_{\mathcal{F}}(0, t; R^m)$:={R^m–valued and predictable stochastic processes such that $E \int_0^t |\phi_s|^p ds < \infty$};
- $D^p_{\mathcal{F}}(0, t; R^m)$:={all RCLL processes in $L^p_{\mathcal{F}}(0, t; R^m)$ such that $E[\sup_{0 \leq s \leq t} |\phi_s|^p] < \infty$};
- $S^p_{\mathcal{F}}(0, T; R^m)$:={all continuous processes in $D^p_{\mathcal{F}}(0, T; R^m)$ };

In the case $m = 1$, we denote them by $L^p(\mathcal{F}_t)$, $L^p_{\mathcal{F}}(0, t)$, $D^p_{\mathcal{F}}(0, t)$ and $S^p_{\mathcal{F}}(0, t)$. We recall that all elements in $D^2_{\mathcal{F}}(0, T)$ are predictable.

For each given $X \in L^2(\mathcal{F}_t)$, we solve the following BSDE on $[0, t]$:

$$Y_s = X + \int_s^t g(r, Y_r, Z_r) dr - \int_s^t Z_r dB_r, \tag{2.4}$$

where the unknown is the pair of the adapted processes (Y, Z). Here the function g is given

$$g : (\omega, t, y, z) \in \Omega \times [0, \infty) \times R \times R^d \to R.$$

It satisfies the following basic assumptions for each $\forall y, y' \in R,\ z, z' \in R^d$

$$\left\{ \begin{array}{l} g(\cdot, y, z) \in L^2_{\mathcal{F}}(0, T), \quad \forall T \in (0, \infty), \\ |g(t, y, z) - g(t, y', z')| \leq \mu(|y - y'| + |z - z'|) . \end{array} \right. \tag{2.5}$$

In some cases it is interesting to consider the following situation:

$$\left\{ \begin{array}{lll} \text{(a)} & g(\cdot, 0, 0) & \equiv 0, \\ \text{(b)} & g(\cdot, y, 0) & \equiv 0, \ \forall y \in R. \end{array} \right. \tag{2.6}$$

Obviously (b) implies (a). This BSDE (2.4) was intrduced by Bismut [5], [6] for the case where g is a linear function of (y, z). Pardoux and Peng [33] obtained the following result (see Theorem 4.3 for a more general situation): for each $X \in L^2(\mathcal{F}_t)$, there exists a unique solution $(Y, Z) \in S^2_{\mathcal{F}}(0, t) \times L^2_{\mathcal{F}}(0, t; R^d)$ of the BSDE (2.4).

Definition 2.9 *We denote by $\mathbb{E}^g_{s,t}[X_t] := Y_s$, $0 \leq s \leq t$. We thus define a system of operators*

$$\mathbb{E}^g_{s,t}[\cdot] : L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s), \quad 0 \leq s \leq t < \infty. \tag{2.7}$$

$(\mathbb{E}^g_{s,t}[\cdot])_{0 \leq s \leq t < \infty}$ is called g–expectation.

Proposition 2.10 *Let the generating function g satisfies (2.5). Then*

$$\mathbb{E}^g_{s,t}[X] : X \in L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s),\ 0 \le s \le t < \infty$$

defined in (2.7) is an $\mathcal{F}_t$–consistent pricing mechanism, called g–pricing mechanism, i.e., it satisfies (A1)–(A4) of Definition 2.1.

This pricing mechanism is entirely generated by function g. We then call g a (contingent claim) price generating function.

Proof. This result is a special case of Proposition 4.9. ■

Since g satisfies Lipschitz condition with Lipschitz constant μ, it is then dominated by the following function

$$g_\mu(y,z) := \mu|y| + \mu|z|,\ (y,z) \in R \times R^d \tag{2.8}$$

in the following since

$$g(t,y,z) - g(t,y',z') \le g_\mu(y-y', z-z').$$

We will see that the above notion of domination is useful. Briefly speaking, a price generating function g is dominated by another one if and only if the corresponding pricing mechanism $\mathbb{E}^g$ is dominated by the other one.

3 Main result: $\mathbb{E}_{s,t}[\cdot]$ is governed by a BSDE

From now on the system $\mathbb{E}_{s,t}[\cdot]_{0\le s\le t<\infty}$ is always a fixed $\mathcal{F}_t$–consistent pricing mechanism, i.e., satisfying (A1)–(A4), with additional assumptions ($A4_0$) and the following $\mathbb{E}^{g_\mu}$–domination assumption:

(A5) there exists a sufficiently large number $\mu > 0$ such that, for each $0 \le s \le t \le T$,

$$\mathbb{E}_{s,t}[X] - \mathbb{E}_{s,t}[X'] \le \mathbb{E}^{g_\mu}_{s,t}[X - X'],\ \ \forall X, X' \in L^2(\mathcal{F}_t), \tag{3.1}$$

where the function $g_\mu(y,z) = \mu|y| + \mu|z|$ is given in (2.8).

The **main theorem** of this paper is:

Theorem 3.1 *We assume that the function g satisfies (2.5) with $g(\cdot,0,0) = 0$. Then the g–expectation $\mathbb{E}^g_{s,t}[\cdot]_{0\le s\le t<\infty}$ is an $\mathcal{F}_t$–consistent pricing mechanism satisfying (A1)–(A4), ($A4_0$) and the domination condition (A5). $\mathbb{E}^g$ is then called g–(contingent claim) pricing mechanism, and the function g is called a (contingent claim) price generating function.*
Conversely, let $\mathbb{E}_{s,t}[\cdot]_{0\le s\le t<\infty}$ be an $\mathcal{F}_t$–consistent pricing mechanism satisfying (A1)–(A4), ($A4_0$) and the domination condition (A5), then there exists a unique price generating function $g(\omega,t,y,z)$ satisfying (2.5) with $g(\cdot,0,0) \equiv 0$, such that

$$\mathbb{E}_{s,t}[X] = \mathbb{E}^g_{s,t}[X],\ \ \forall s \le t,\ \forall X \in L^2(\mathcal{F}_t). \tag{3.2}$$

Remark 3.2 *The case where $\mathbb{E}_{s,t}[\cdot]$ satisfy (A1)–(A5), without (A4$_0$), can be obtained as corollaries of the this main theorem. In this more general situation the condition $g(s,0,0) \equiv 0$ is not imposed. The main result of [11]*

We consider some special situations of our theorem.

Example 3.3 *If moreover, $g(s,y,0) \equiv 0$. Then, by [36], (A2') holds. Thus, according to Proposition 2.5, $\mathbb{E}^g_{s,t}[\cdot]$ becomes an $\mathcal{F}_t$–consistent nonlinear expectation:*

$$\mathbb{E}[X|\mathcal{F}_t] = \mathbb{E}_g[X|\mathcal{F}_t] := \mathbb{E}^g_{s,t}[X] = \mathbb{E}^g_{s,T}[X].$$

This is so called g–expectation introduced in [36].

This extends non trivially the result obtained in [11], (see also [41] for a more systematical presentation and explanations in finance), where we needed a more strict domination condition plus the following assumption

$$\mathbb{E}[X+\eta|\mathcal{F}_t] = \mathbb{E}[X|\mathcal{F}_t] + \eta,\ \forall \eta \in \mathcal{F}_t.$$

Under these assumptions we have proved in [11] that there exists a unique function $g = g(s,z)$, with $g(s,0) \equiv 0$, such that $\mathbb{E}_g[X] \equiv \mathbb{E}[X] = \mathbb{E}[X|\mathcal{F}_0]$.

Example 3.4 *Consider a financial market consisting of $d+1$ assets: one bond and d stocks. We denote by $P_0(t)$ the price of the bond and by $P_i(t)$ the price of the i-th stock at time t. We assume that P_0 is the solution of the ordinary differential equation: $dP_0(t) = r(t)P_0(t)dt$, and $\{P_i\}_{i=1}^d$ is the solution of the following SDE*

$$dP_i(t) = P_i(t)[b_i(t)dt + \sum\nolimits_{j=1}^d \sigma_{ij}(t)dB_t^j],$$
$$P_i(0) = p_i, \quad i = 1, \cdots, d.$$

Here r is the interest rate of the bond; $\{b_i\}_{i=1}^d$ is the rate of the expected return, $\{\sigma_{ij}\}_{i,j=1}^d$ the volatility of the stocks. We assume that r, b, σ and σ^{-1} are all $\mathcal{F}_t$–adapted and uniformly bounded processes on $[0,\infty)$. Black and Scholes have solved the problem of the market pricing mechanism of an European type of derivative $X \in L^2(\mathcal{F}_T)$ with maturity T. In the point of view of BSDE, the problem can be treated as follows: consider an investor who has, at a time $t \le T$, $n_0(t)$ bonds and $n_i(t)$ i-stocks, $i = 1, \cdots, d$, i.e., he invests $n_0(t)P_0(t)$ in bond and $\pi_i(t) = n_i(t)P_i(t)$ in the i-th stock. $\pi(t) = (\pi_1(t), \cdots, \pi_d(t))$, $0 \le t \le T$ is an R^d valued, square-integrable and adapted process. We define by $y(t)$ the investor's wealth invested in the market at time t:

$$y(t) = n_0(t)P_0(t) + \sum\nolimits_{i=1}^d \pi_i(t).$$

We make the so called self–financing assumption: in the period $[0,T]$, the investor does not withdraw his money from, or put his money in his account y_t. Under this condition, his wealth $y(t)$ evolves according to

$$dy(t) = n_0(t)dP_0(t) + \sum\nolimits_{i=1}^d n_i(t)dP_i(t).$$

or

$$dy(t) = [r(t)y(t) + \sum\nolimits_{i=1}^{d}(b_i(t) - r(t))\pi_i(t)]dt + \sum\nolimits_{i,j=1}^{d}\sigma_{ij}(t)\pi_i(t)dB_t^j.$$

We denote $g(t,y,z) := -r(t)y - \sum_{i,j=1}^{d}(b_i(t) - r(t))\sigma_{ij}^{-1}(t)z_j$. *Then, by the variable change* $z_j(t) = \sum_{i=1}^{d}\sigma_{ij}(t)\pi_i(t)$, *the above equation is*

$$-dy(t) = g(t, y(t), z(t))dt - z(t)dB_t.$$

We observe that function g satisfies (2.5). It follows from the existence and uniqueness theorem of BSDE (Theorem 4.3) that for each derivative $X \in L^2(\mathcal{F}_T)$, there exists a unique solution $(y(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(0,T;R^{1+d})$ with the terminal condition $y_T = X$. This meaning is significant: in order to replicate the derivative X, the investor needs and only needs to invest $y(t)$ at the present time t and then, during the time interval $[t,T]$ and then to perform the portfolio strategy $\pi_i(s) = \sigma_{ij}^{-1}(s)z_j(s)$. Furthermore, by Comparison Theorem of BSDE, if he wants to replicate a X' which is bigger than X, (i.e., $X' \geq X$, a.s., $P(X' \geq X) > 0$), then he must pay more, i.e., there this no arbitrage opportunity. This $y(t)$ is called the Black–Scholes price, or Black–Scholes pricing mechanism, of X at the time t. We define, as in (4.6), $\mathbb{E}^g_{t,T}[X] = y_t$. We observe that the function g satisfies (b) of condition (2.6). It follows from Proposition 4.9 that $\mathbb{E}^g_{t,T}[\cdot]$ satisfies properties (A1)–(A4) for $\mathcal{F}_t$–consistent pricing mechanism.

Example 3.5 *An very important problem is: if we know that the pricing mechanism of an investigated agent is a g–pricing mechanism $\mathbb{E}^g$, how to find this price generating function g. We now consider a case where g depends only on z, i.e., $g = g(z) : \mathbf{R}^d \to \mathbf{R}$. In this case we can find such g by the following testing method. Let $\bar{z} \in \mathbf{R}^d$ be given. We denote $Y_s := \mathbb{E}^g_{s,T}[\bar{z}(B_T - B_t)]$, $s \in [t,T]$, where t is the present time. It is the solution of the following BSDE*

$$Y_s = \bar{z}(B_T - B_t) + \int_s^T g(Z_u)du - \int_s^T Z_u dB_u,\ s \in [t,T].$$

It is seen that the solution is $Y_s = \bar{z}(B_s - B_t) + \int_s^T g(\bar{z})ds$, $Z_s \equiv \bar{z}$. Thus

$$\mathbb{E}^g_{t,T}[\bar{z}(B_T - B_t)] = Y_t = g(\bar{z})(T-t),$$

or

$$g(\bar{z}) = (T-t)^{-1}\mathbb{E}^g_{t,T}[\bar{z}(B_T - B_t)]. \tag{3.3}$$

Thus the function g can be tested as follows: at the present time t, we ask the investigated agent to evaluate $\bar{z}(B_T - B_t)$. We thus get $\mathbb{E}^g_{t,T}[\bar{z}(B_T - B_t)]$. Then $g(\bar{z})$ is obtained by (3.3).

Remark 3.6 *The above test works also for the case $g : [0,\infty) \times \mathbf{R}^d \to \mathbf{R}$, or for a more general situation $g = \gamma y + g_0(t,z)$.*

An interesting problem is, in general, how to find the price generating function g by a testing of the input–output behavior of $\mathbb{E}^g[\cdot]$? Let $b : R^n \longmapsto R^n$, $\bar{\sigma} : R^n \longmapsto R^{n\times d}$ be two Lipschitz functions.

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dB_r, \;\; s \geq t.$$

The following result was obtained in Proposition 2.3 of [7].

Proposition 3.7 *We assume that the price generating function g satisfies (2.5). We also assume that, for each fixed (y,z), $g(\cdot,y,z) \in D^2_{\mathcal{F}}(0,T)$. Then for each $(t,x,p,y) \in [0,\infty) \times R^n \times R^n \times R$, we have*

$$L^2\text{–}\lim_{\epsilon\to 0} \frac{1}{\epsilon}[\mathbb{E}^g_{t,t+\epsilon}[y + p\cdot(X^{t,x}_{t+\epsilon} - x)] - y] = g(t,y,\sigma^T(x)p) + p\cdot b(x).$$

4 Pricing an accumulated contingent claim with $\mathbb{E}^g$–pricing mechanisms

Definition 4.1 *An accumulated contingent claim $(X,K) \in L^2(\mathcal{F}_T) \times D^2_{\mathcal{F}}(0,T)$ with maturity T is a contract, according which the writer have to pay the buyer X at T and. in each time interval $[s,t] \subset [0,T]$, $K_t - K_s$.*

Remark 4.2 *We understand that, in a real life, X should be non negative and K non decreasing. But we will see that we can treat the general situation $(X,K) \in L^2(\mathcal{F}_T) \times D^2_{\mathcal{F}}(0,T)$, without any mathematical obstacle.*

We consider the following BSDE on $[0,t]$ with given terminal condition $X \in L^2(\mathcal{F}_t)$ and an RCLL process $K \in D^2_{\mathcal{F}}(0,\infty)$:

$$Y_s = X + K_t - K_s + \int_s^t g(r,Y_r,Z_r)dr - \int_s^t Z_r dB_r, \; s \in [0,t]. \tag{4.1}$$

When K is an increasing (resp. decreasing) process, the solution (Y,Z) is called a g–supersolution (resp. g–subsolution). These type of solutuions appear very often in superhedging problem in the pricing of contingent claims in an incomplete markets, where one need to find the smallest g–supersolutuion (resp. the largest g–subsolution) to replicate X. We first recall the following basic results of BSDE.

Theorem 4.3 *([33], [35]) We assume (2.5). Then there exists a unique solution $(Y,Z) \in L^2_{\mathcal{F}}(0,t;R\times R^d)$ of BSDE (4.1). We denote it by*

$$(Y_s^{t,X,K}, Z_s^{t,X,K}) = (Y_s, Z_s), \; s \in [0,t]. \tag{4.2}$$

We have

$$Y^{t,X,K} + K \in S^2_{\mathcal{F}}(0,t).$$

Proof. In [33] (see also [24]), the result of BSDE is for $t = T$and $K_t = \int_0^t \phi_s ds$ for some $\phi \in L^2_{\mathcal{F}}(0,T)$. The present situation can be treated by setting (see [36])

$$\begin{aligned} \bar{Y}_s &:= Y_s + K_s, \\ \bar{g}(s,y,z) &:= g(s, y - K_s, z)1_{[0,t]}(s) \end{aligned} \tag{4.3}$$

and considering the following equivalent BSDE

$$\bar{Y}_s = X + K_t + \int_s^T \bar{g}(r, \bar{Y}_r, Z_r)dr - \int_s^T Z_r dB_r,\ s \in [0,T]. \tag{4.4}$$

It is clear that $\bar{Y}_s \equiv X + K_s$, $Z_s \equiv 0$on $[t,T]$. Since $\bar{g}$ is a Lipschitz function with the same Lipschitz constant $\mu$and

$$\bar{g}(\cdot, 0, 0) = g(\cdot, -K_\cdot, 0)1_{[0,t]}(\cdot) \in L^2_{\mathcal{F}}(0,T),$$

thus, by [33], [35], the BSDE (4.4) has a unique solution $(\bar{Y}, Z)$. ■

We introduce a new notation.

Definition 4.4 *We denote, for $s \le t$,*

$$\mathbb{E}^g_{s,t}[X; K_\cdot] := Y^{t,X,K}_s \tag{4.5}$$

$$\mathbb{E}^g_{s,t}[X] := \mathbb{E}^g_{s,t}[X; 0]. \tag{4.6}$$

This notion generalizes that of $\mathbb{E}^g_{s,t}[\cdot]$ in Definition 2.9. Clearly when (2.6)–(a) is satisfied, we have $\mathbb{E}^g_{s,t}[0] = \mathbb{E}^g_{s,t}[0; 0] = 0$, $0 \le s \le t \le T$.

Remark 4.5 *In fact, for each maturity $T \ge 0$ the price process of the accumulated contingent claim $(X, K) \in L^2(\mathcal{F}_T) \times D^2_{\mathcal{F}}(0,T)$ produced by $\mathbb{E}^g[\cdot]$ is $\mathbb{E}^g_{s,T}[X; K]$, $s \le T$. We will prove it for a more general price mechanism $\mathbb{E}[\cdot]$, see the next subsection.*

Remark 4.6 *About the notations $\mathbb{E}^g[\cdot]$. This notation was firstly introduced in [36] in the case where g satisfies (2.6)–(b). In this situation it is easy to check that*

$$\mathbb{E}^g_{s,t}[X] \equiv \mathbb{E}^g_{s,T}[X],\ \forall 0 \le s \le t \le T.$$

In other words, $\mathbb{E}^g$–is a nonlinear expectation, called g–expectation. The general situation, i.e., without (2.6) was introduced in [35] and [16].

By the above existence and uniqueness theorem, we have for each $0 \le r \le s \le t$ and for each $X \in L^2(\mathcal{F}_t)$ and $K \in D^2_{\mathcal{F}}(0,T)$,

$$\mathbb{E}^g_{r,s}[\mathbb{E}^g_{s,t}[X; K_\cdot]; K_\cdot] = \mathbb{E}^g_{r,t}[X; K_\cdot],\ \text{a.s.} \tag{4.7}$$

It is also easy to check that, with the notation $g_-(t,y,z) := -g(t,-y,-z)$

$$-\mathbb{E}^g_{s,t}[X; K_\cdot] = \mathbb{E}^{g_-}_{s,t}[-X; -K_\cdot]. \tag{4.8}$$

We will see that $\{\mathbb{E}^g_{t,T}[X]\}_{0 \le t \le T}$, $X \in L^2(\mathcal{F}_T)$ form an $\mathcal{F}_t$–consistent nonlinear pricing mechanism. The following monotonicity property is the comparison theorem of BSDE.

Theorem 4.7 *We assume (2.5). For each fixed maturity let for let (X,K) and $(X',K') \in L^2(\mathcal{F}_t)\times D^2_{\mathcal{F}}(0,T)$ be two accumulated contingent claims satisfying $X\geq X'$ and that $K-K'$ is an increasing process. Then we have*

$$\mathbb{E}^g_{s,t}[X;K_\cdot]\geq \mathbb{E}^g_{s,t}[X';K'_\cdot],\quad \forall s\leq t. \tag{4.9}$$

In particular,

$$\mathbb{E}^g_{s,t}[X]\geq \mathbb{E}^g_{s,t}[X']. \tag{4.10}$$

If $A\in D^2_{\mathcal{F}}(0,T)$ is an increasing process, then

$$\mathbb{E}^g_{s,t}[X;A_\cdot]\geq \mathbb{E}^g_{s,t}[X]. \tag{4.11}$$

Proof. The case $K_t\equiv K'_t\equiv 0$ is the classical comparison theorem of BSDE. The present general situation, see [35] or [41]. ■

We recall the special price generating function $g_\mu(y,z)$ defined in (2.8). It is a very strong generating function. In fact we have

Corollary 4.8 *The g–pricing mechanism $\mathbb{E}^g$ is dominated by $\mathbb{E}^{g_\mu}$ in the following sense: for each $t\geq 0$, let (X,K) and $(X',K')\in L^2(\mathcal{F}_t)\times D^2_{\mathcal{F}}(0,T)$ be two accumulated contingent claims with maturity t, then we have*

$$\mathbb{E}^g_{s,t}[X;K_\cdot]-\mathbb{E}^g_{s,t}[X';K'_\cdot]\leq \mathbb{E}^{g_\mu}_{s,t}[X-X';K_\cdot-K'_\cdot] \tag{4.12}$$

where μ is the Lipschitz constant of g given in (2.5). In particular, since g_μ the generating function g_μ itself has Lipschitz constant μ, we have

$$\mathbb{E}^{g_\mu}_{s,t}[X;K_\cdot]-\mathbb{E}^{g_\mu}_{s,t}[X';K'_\cdot]\leq \mathbb{E}^{g_\mu}_{s,t}[X-X';K_\cdot-K'_\cdot] \tag{4.13}$$

Proof. By the definition, The pricing processes produces by $Y_s=\mathbb{E}^g_{s,t}[X;K_\cdot]$ and $Y'_s=\mathbb{E}^g_{s,t}[X';K'_\cdot]$ solve respectively the following BSDEs on $[0,t]$:

$$Y_s = X+K_t-K_s+\int_s^t g(r,Y_r,Z_r)dr-\int_s^t Z_r dB_r,$$
$$Y'_s = X'+K'_t-K'_s+\int_s^t g(r,Y'_r,Z'_r)dr-\int_s^t Z'_r dB_r.$$

We denote $\hat{Y}=Y-Y'$, $\hat{Z}=Z-Z'$ and

$$\hat{K}_t=K_t-K'_t+\int_0^t[-g_\mu(\hat{Y}_s,\hat{Z}_s)+g(s,Y_s,Z_s)-g(s,Y_s,Z_s)]ds.$$

Then $(\hat{Y},\hat{Z})$ solves a new BSDE

$$\hat{Y}_s=X-X'+\hat{K}_t-\hat{K}_s+\int_s^t g_\mu(r,\hat{Y}_r,\hat{Z}_r)dr-\int_s^t \hat{Z}_r dB_r.$$

We compare it to the BSDE

$$\bar{Y}_s=X-X'+(K-K')_t-(K-K')_s+\int_s^t g_\mu(r,\bar{Y}_r,\bar{Z}_r)dr-\int_s^t \bar{Z}_r dB_r.$$

Since $d(K - K' - \hat{K})_s \geq 0$, thus, by comparison theorem, i.e., Theorem 4.7, $\bar{Y}_s \geq \hat{Y}_s = Y_s - Y'_s$. We thus have (4.12). ■

It is very interesting to observe that, given a price generating function g, $\mathbb{E}^g_{s,t}[\cdot; K.]$ is again an $\mathcal{F}_t$–consistent pricing mechanism:

Proposition 4.9 *Let the generating function g satisfies (2.5) and for a fixed $K \in D^2_{\mathcal{F}}(0,\infty)$,*

$$\mathbb{E}^g_{s,t}[X; K.] : X \in L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s),\ 0 \leq s \leq t < \infty \tag{4.14}$$

defined in (4.5) is an $\mathcal{F}_t$–consistent pricing mechanism, i.e., it satisfies (A1)–(A4) of Definition 2.1.

Proof. (A1) is given by (4.9). (A2) is clearly true by the definition. (A3) is proved by (4.7). We now consider (A4), i.e., for each t and $X \in L^2(\mathcal{F}_t)$, we have

$$1_A \mathbb{E}^g_{s,t}[X; K.] = 1_A \mathbb{E}^g_{s,t}[1_A X; K.],\ \ \forall s \leq t,\ \ A \in \mathcal{F}_t. \tag{4.15}$$

as well as

$$1_A \mathbb{E}^g_{s,t}[X; K.] = \mathbb{E}^{g_{s,A}}_{s,t}[1_A X; K^{s,A}_{\cdot}],\ \ \forall A \in \mathcal{F}_t, \tag{4.16}$$

where we set

$$g_{s,A}(t,y,z) := 1_{[0,s)}(t) g(t,y,z) + 1_{[s,T]}(t) 1_A g(t,y,z), \tag{4.17}$$

$$K^{s,A}_t := 1_{[0,s)}(t) K_t + 1_{[s,T]}(t) 1_A (K_t - K_s). \tag{4.18}$$

We will give the proof of (4.15). The proof of (4.16) is similar. According to BSDE (4.1) for each time $r \in [s,t]$, $Y_r := \mathbb{E}^g_{r,s}[X; K.]$ and $\bar{Y}_r := \mathbb{E}^g_{s,t}[1_A X; K.]$ solve respectively

$$Y_r = X + K_t - K_r + \int_r^t g(r, Y_u, Z_u) du - \int_r^t Z_u dB_u,$$

and

$$\bar{Y}_r = 1_A X + K_t - K_r + \int_r^t g(u, \bar{Y}_u, \bar{Z}_u) du - \int_r^t \bar{Z}_u dB_u$$

We multiply 1_A, $A \in \mathcal{F}_s$ on both sides of the above two BSDEs. Since $1_A g(r, Y_r, Z_r) = 1_A g(r, Y_r 1_A, Z_r 1_A)$, we have

$$1_A Y_r = 1_A X + 1_A K_t - 1_A K_r + \int_r^t 1_A g(u 1_A Y_u, 1_A Z_u) du - \int_r^t 1_A Z_u dB_u,$$

and

$$1_A \bar{Y}_r = 1_A X + 1_A K_t - 1_A K_r + \int_r^t 1_A g(u 1_A Y_u, 1_A Z_u) du - \int_r^t 1_A \bar{Z}_u dB_u.$$

It is clear that $1_A Y_r$ and $1_A \bar{Y}_r$ satisfy exactly the same BSDE with the same terminal condition on $[s,t]$. By uniqueness of BSDE, $1_A Y_r \equiv 1_A \bar{Y}_r$ on $[s,t]$, i.e., $1_A \mathbb{E}^g_{s,t}[X; K.] \equiv 1_A \mathbb{E}^g_{s,t}[1_A X; K.]$. The proof is complete. ■

If Y is the data of a price process produced by some contingent claim pricing mechanism, in many situations it is practically meaningful and financially interesting to compare this data by using a given g–pricing mechanism. One typical situation is that the price produced by $\mathbb{E}^g$ is weaker (resp. stronger). In this situation Y is called a g–supermartingale (resp. g–submartingale). Here the term "g–martingale" is a nonlinear, and nontrivial generalization of the classical one, due to the similarity between the classical conditional expectation $\mathbf{E}[\cdot|\mathcal{F}_s]$ and $\mathbb{E}^g_{s,t}[\cdot]$: .

Definition 4.10 *Let $K \in D^2_{\mathcal{F}}(0,\infty)$ be given. A process $Y \in D^2_{\mathcal{F}}(0,\infty)$ is said to be an $\mathbb{E}^g[\cdot;K]$–martingale (resp. $\mathbb{E}^g[\cdot;K]$–supermartingale, $\mathbb{E}^g[\cdot;K]$–submartingale) if for each $0 \le s \le t$*

$$\mathbb{E}^g_{s,t}[Y_t;K.] = Y_s, \ (\textit{resp.} \ \le Y_s, \ \ge Y_s). \tag{4.19}$$

Clearly a $\mathbb{E}^g$–martingale Y is a price process produced by this pricing mechanism: $Y_s = \mathbb{E}^g_{s,T}[Y_T]$, $s \le T$.

Remark 4.11 *If $(y,z) \in L^2_{\mathcal{F}}(0,T;R\times R^d)$ solves the BSDE*

$$y_s = y_t + K_t - K_s + \int_s^t g(r,y_r,z_r)dr - \int_s^t z_r dB_r, \ s \le t.$$

It is clear that $(-y,-z)$ solves

$$\begin{aligned}-y_s = & -y_t + (-K_t) - (-K_s) \\ & + \int_s^t [-g(r,-(-y_r),-(-z_r))dr - \int_s^t (-z_r)dB_r.\end{aligned}$$

Thus, if y is an $\mathbb{E}^g[\cdot;K.]$–martingale (resp. $\mathbb{E}^g[\cdot;K]$–supermartingale, $\mathbb{E}^g[\cdot;K]$–submartingale), then $-y$ is an $\mathbb{E}^{g_*}_{s,t}[\cdot;-K.]$–martingale (resp. $\mathbb{E}^{g_*}[\cdot;K]$–submartingale, $\mathbb{E}^{g_*}[\cdot;K]$–supermartingale), where we denote

$$g_*(t,y,z) := -g(t,-y,-z).$$

Therefor many results concerning $\mathbb{E}^g[\cdot;K]$–supermartingales can be also applied to situations of submartingales.

Example 4.12 *Let $X \in L^2(\mathcal{F}_T)$ and $A \in D^2_{\mathcal{F}}(0,T)$ be given such that A is an increasing process. By the monotonicity of $\mathbb{E}^g$, i.e., Theorem 4.7, we have, for $t \in [0,T]$,*

$$\begin{aligned}&Y_t := \mathbb{E}^g_{t,T}[X] = \mathbb{E}^g_{t,T}[X;0] \ \textit{is a } \mathbb{E}^g\textit{–martingale,}\\ &Y_t^+ := \mathbb{E}^g_{t,T}[X;A] \ \textit{is a } \mathbb{E}^g\textit{–supermartingale,}\\ &Y_t^- := \mathbb{E}^g_{t,T}[X;-A] \ \textit{is a } \mathbb{E}^g\textit{–submartingale.}\end{aligned}$$

As in classical situations, an interesting and hard problem is the inverse one: if Y is an $\mathbb{E}^g$–supermartingale, can we find an increasing and predictable process A such that $Y_t \equiv \mathbb{E}^g_{t,T}[X;A]$? This nonlinear version of Doob–Meyer's

decomposition theorem will be stated as follows. It plays a crucially important role in this paper.

We have the following $\mathbb{E}^g$–supermartingale decomposition theorem of Doob–Meyer's type. This nonlinear decomposition theorem was obtained in [38]. But the formulation using the notation $\mathbb{E}^g_{t,T}[\cdot; A]$ is new. In fact we think this is the intrinsic formulation since it becomes necessary in the more abstract situation of the $\mathbb{E}$–supermartingale decomposition theorem, i.e., Theorem 8.1 which can considered as a generalization of the following result.

Proposition 4.13 *We assume (2.5)–(i) and (ii). Let $Y \in D^2_{\mathcal{F}}(0,T)$ be an $\mathbb{E}^g$–supermartingale. Then there exists a unique increasing process $A \in D^2_{\mathcal{F}}(0,T)$ (thus predictable) with $A_0 = 0$, such that*

$$Y_t = \mathbb{E}^g_{t,T}[Y_T; A], \ \forall 0 \le t \le T. \tag{4.20}$$

Corollary 4.14 *Let $K \in D^2_{\mathcal{F}}(0,T)$ be given and let $Y \in D^2_{\mathcal{F}}(0,T)$ be an $\mathbb{E}^g[\cdot; K]$–supermartingale in the following sense*

$$\mathbb{E}^g_{s,t}[Y_t; K] \le Y_s, \ \forall 0 \le s \le t \le T. \tag{4.21}$$

Then there exists a unique increasing process $A \in D^2_{\mathcal{F}}(0,T)$ with $A_0 = 0$, such that

$$Y_t = \mathbb{E}^g_{t,T}[Y_T; K + A], \ \forall 0 \le t \le T. \tag{4.22}$$

Proof. By the notations of (4.3) with $t = T$, we have

$$\mathbb{E}^g_{s,t}[Y_t; K] + K_s = \mathbb{E}^{\bar{g}}_{s,t}[Y_t + K_t]. \tag{4.23}$$

It follows that (4.21) is equivalent to

$$\mathbb{E}^{\bar{g}}_{s,t}[Y_t + K_t] \le Y_s + K_s, \ \forall 0 \le s \le t \le T. \tag{4.24}$$

In other words, $Y + K$ is an $\mathbb{E}^{\bar{g}}$–supermartingale in the sense of (4.19). By the above supermartingale decomposition theorem, Proposition 4.13, there exists an increasing process $A \in D^2_{\mathcal{F}}(0,T)$ with $A_0 = 0$, such that

$$Y_t + K_t = \mathbb{E}^{\bar{g}}_{t,T}[Y_T + K_T; A], \ \forall 0 \le t \le T, \tag{4.25}$$

or, equivalently (4.22). ■

5 Characterization of g-pricing mechanism by its generating function g

For a price mechanism $\mathbb{E}^g[\cdot]$, it is important to distinct her selling price and buying price. We now fix that $\mathbb{E}^g[\cdot]$ is the selling price. A rational price mechanism must be

$$\mathbb{E}^g[X] \ge -\mathbb{E}^g[-X].$$

It also possesses some other properties, such as convexity, or moreover, subadditivity. See [2], [3], [4], [9], [23], [24], [26], [28], [37], [45], etc. for the ecomomic meanings. An interesting question is: what the corresponding generating function g will behaves if the the $\mathbb{E}^g$ satisfies the above properties. We will see that g perfectly reflects the behavior of $\mathbb{E}^g$. This will be very important for using data of the pricing processes to statistically find g. We begin with introducing some technique lemmas.

Let a functions $f : (\omega, t, y, z) \in \Omega \times [0, T] \times R \times R^d \to R$ satisfy the same Lipschitz condition (2.5) as for g. For each $n = 1, 2, 3, \cdots$, we set

$$f^n(s, y, z) := \sum_{i=0}^{2^n-1} f(s, Y_s^{t_i^n, y}, z) 1_{[t_i^n, t_{i+1}^n)}(s), \ s \in [0, T] \tag{5.1}$$

$$t_i^n = i2^{-n}T, i = 0, 1, 2, \cdots, 2^n. \tag{5.2}$$

It is clear that f^n is an $\mathcal{F}_t$–adapted process.

For each fixed $(t, y, z) \in [0, T] \times R \times R^d$, we consider the following SDE of Itô's type defined on $[t, T]$:

$$Y_s^{t,y,z} = y - \int_t^s f(r, Y_r^{t,y,z}, z)dr + z(B_s - B_t) \tag{5.3}$$

We have the following classical result of Itô's SDE.

Lemma 5.1 *We assume that f satisfies the same Lipschitz condition (2.5) as for g. (2.5). Then there exists a constant C, depending only on μ, T and $E\int_0^T |f(\cdot, 0, 0)|^2 ds$, such that, for each $(t, y, z) \in [0, T] \times R \times R^d$, we have*

$$E[|Y_s^{t,y,z} - y|^2] \leq C(|y|^2 + |z|^2 + 1)(s - t), \ \forall s \in [t, T]. \tag{5.4}$$

Proof. It is classic that $E\int_0^T |f(r, Y_r^{t,y,z}, z)|dr^2 \leq C_0(|y|^2 + |z|^2 + 1)$, where C_0 depends only on μ, T and $E\int_0^T |f(\cdot, 0, 0)|^2 ds$. We then have

$$\begin{aligned}
E[|Y_s^{t,y,z} - y|^2] &\leq 2E[|\int_t^s f(r, Y_r^{t,y,z}, z)dr|^2] + 2|z|^2(s - t) \\
&\leq 2E[|\int_t^s f(r, Y_r^{t,y,z}, z)dr|^2] + 2|z|^2(s - t) \\
&\leq 2E[\int_t^s |f(r, Y_r^{t,y,z}, z)|^2 dr](t - s) + 2|z|^2(s - t) \\
&\leq C(|y|^2 + |z|^2 + 1)(s - t).
\end{aligned}$$

■

Lemma 5.2 *For each fixed $(y, z) \in R \times R^d$, $\{f^n(\cdot, y, z)\}_{n=1}^\infty$ converges to $f(\cdot, y, z)$ in $L^2_{\mathcal{F}}(0, T)$, i.e.,*

$$\lim_{n\to\infty} E\int_0^T |f^n(s, y, z) - f(s, y, z)|^2 ds = 0. \tag{5.5}$$

Proof. For each $s \in [0,T)$, there are some integers $i \leq 2^n - 1$ such that $s \in [t_i^n, t_{i+1}^n)$. We have, by (5.4)

$$\begin{aligned} E[|f^n(s,y,z) - f(s,y,z)|^2] &= E[|f(s, Y_s^{t_i^n,y}, z) - f(s,y,z)|^2] \\ &\leq \mu^2 E[|Y_s^{t_i^n,y,z} - y|^2] \\ &\leq \mu^2 C(|y|^2 + |z|^2 + 1)2^{-n}T. \end{aligned}$$

Thus $\{f^n(\cdot,y,z)\}_{n=1}^{\infty}$ converges to $f(\cdot,y,z)$ in $L^2_{\mathcal{F}}(0,T)$. ■

Lemma 5.3 *Let $f : (\omega,t,y,z) \in \Omega \times [0,T] \times R \times R^d \to R$ satisfies the same Lipschitz condition (2.5) as for g. If for each $(t,y,z) \in [0,T] \times R \times R^d$, we have*

$$f(\omega, r, Y_r^{t,y,z}, z) \geq 0 \quad \textit{(resp. } = 0\textit{)},\ (\omega, r) \in [t,T] \times \Omega,\ .dr \times dP\text{-a.s.}.$$

Then We then, for each $(y,z) \in R \times R^d$

$$f(\omega,t,y,z) \geq 0,\ \textit{(resp. } = 0\textit{)},\ (\omega,t) \in [0,T] \times \Omega,\ .dt \times dP\text{-a.s.}. \tag{5.6}$$

Proof. Let us fix y and z. We define $f^n(s,y,z)$ as in (5.1). It is clear that,

$$f^n(r,y,z) \geq 0,\ (\text{resp. } = 0), (\omega, r) \in [0,T] \times \Omega,\ .dr \times dP \text{ a.s.}$$

But from Lemma 5.2 we have $f^n(\cdot,y,z) \to f(\cdot,y,z)$, in $L^2_{\mathcal{F}}(0,T)$ as $n \to \infty$. We thus have **5.6**. ■

We need the following inverse comparison theorem which generalizes the results of [7] and [10] in the sense that g does not need to be continuous, or right continuous, in time. We thus finally obtain an equivalent conditions under the standard condition (2.5) of BSDE. We notice that this result was obtained by already by [30] and [31]. Here we will use a very different method that will be applied in the proof of our main theorem.

Proposition 5.4 *Let g, $\bar{g} : (\omega,t,y,z) \in \Omega \times [0,T] \times R \times R^d \to R$ be two price generating functions satisfying (2.5). Then the following two conditions are equivalent:*
(i) *$g(\omega,t,y,z) \geq \bar{g}(\omega,t,y,z)$, $\forall (y,z) \in R \times R^d$, $dP \times dt$ a.s.*
(ii) *The corresponding pricing mechanisms $\mathbb{E}^g[\cdot]$, $\mathbb{E}^{\bar{g}}[\cdot]$ satisfy*

$$\mathbb{E}^g_{s,t}[\xi] \geq \mathbb{E}^{\bar{g}}_{s,t}[\xi],\ \forall 0 \leq s \leq t,\ \forall \xi \in L^2(\mathcal{F}_t)$$

Proof. The method of the proof is significantly different from [7] and [10].
The part **(i)⇒(ii)** is simply from the standard comparison theorem of BSDE. We now prove the part **(ii)⇒(i).** For each fixed $(t,y,z) \in [0,T] \times R \times R^d$, the solution $(Y_s^{t,y,z})_{s\in[t,T]}$ of SDE with $f = g$ is a $\bar{g}$–supermartingale. By the decomposition theorem, there exists an increasing process $A \in D^2_{\mathcal{F}}(0,T)$ such that

$$Y_s^{t,y,z} = y - \int_t^s \bar{g}(r, Y_r^{t,y,z}, z)dr - (A_s - A_t) + z(B_s - B_t),$$

Comparing this with $Y_s^{t,y,z} = y - \int_t^s g(r, Y_r^{t,y,z}, z)dr + z(B_s - B_t)$, we have

$$g(r, Y_r^{t,y,z}, z) \geq \bar{g}(r, Y_r^{t,y,z}, z), \text{ a.e, in } [t,T], \text{ a.s..}$$

We then can apply Lemma 5.3 to obtain (i). ■

Corollary 5.5 *The following two conditions are equivalent:*
(i) *The price generating function g satisfies. for each $(y,z) \in R \times R^d$,*

$$g(t,y,z) \geq -g(t,-y,-z), \text{ a.e., a,s,,}$$

(ii) $\mathbb{E}_{s,t}^g[\cdot] : L^2(\mathcal{F}_t) \longmapsto L^2(\mathcal{F}_s)$ *is a seller's pricing mechanism, i.e., for each* $0 \leq s \leq t$, $\mathbb{E}_{s,t}^g[\xi] \geq -\mathbb{E}_{s,t}^g[-\xi]$, *for each* $\xi \in L^2(\mathcal{F}_t)$.

Proof. We denote $\bar{g}(t,y,z) := -g(t,-y,-z)$ and compare the following two BSDE:

$$Y_s = \xi + \int_s^t g(r, Y_r, Z_r dr - \int_s^t Z_r dB_r, \ s \in [0,t],$$

and

$$\bar{Y}_s = \xi + \int_s^t \bar{g}(r, \bar{Y}, \bar{Z}_r)dr - \int_s^t \bar{Z}_r dB_r, \ s \in [0,t].$$

By the above Proposition, one has $\mathbb{E}_{s,t}^g[\cdot] \geq \mathbb{E}_{s,t}^{\bar{g}}[\cdot]$, iff $g \geq \bar{g}$. This with $\mathbb{E}_{s,t}^{\bar{g}}[\xi] = -\mathbb{E}_{s,t}^g[-\xi]$ yields (i) $\Leftrightarrow$ (ii). ■

Proposition 5.6 *The following two conditions are equivalent:*
(i) *The price generating function $g = g(t,y,z)$ is convex (resp. concave) in (y,z), i.e., for each (y,z) and (y',z') in $R \times R^d$ and for a.e. $t \in [0,T]$*

$$\begin{aligned} g(s, \alpha y + (1-\alpha)y', \alpha z + (1-\alpha)z') &\leq \alpha g(s,y,z) + (1-\alpha)g(s,y',z'), \text{ a.s.} \\ &(\text{resp.} \geq \alpha g(s,y,z) + (1-\alpha)g(s,y',z'), \text{ a.s.}). \end{aligned}$$

(ii) *The corresponding pricing mechanism $\mathbb{E}_{s,t}^g[\cdot]$ is a convex (resp. concave), i.e., for each fixed $\alpha \in [0,1]$, we have*

$$\begin{aligned} \mathbb{E}_{s,t}^g[\alpha\xi + (1-\alpha)\zeta] &\leq \alpha \mathbb{E}_{s,t}^g[\xi] + (1-\alpha)\mathbb{E}_{s,t}^g[\zeta], \text{ a.s.} \\ (\text{resp. } &\geq \alpha \mathbb{E}_{s,t}^g[\xi] + (1-\alpha)\mathbb{E}_{s,t}^g[\zeta], \text{ a.s.}) \\ &\forall s \leq t, \ \forall \xi, \zeta \in L^2(\mathcal{F}_t). \end{aligned} \tag{5.7}$$

Proof. We only prove the convex case.
(i)$\Rightarrow$(ii): For a given $t > 0$, we set $Y_s^\xi := \mathbb{E}_{s,t}^g[\xi]$, $Y_s^\zeta := \mathbb{E}_{s,t}^g[\zeta]$, $s \in [0,t]$. These two pricing processes solve respectively the following two BSDEs on $[0,t]$:

$$\begin{aligned} Y_s^\xi &= \xi + \int_s^t g(r, Y_r^\xi, Z_r^\xi)dr - \int_s^t Z_r^\xi dB_r, \\ Y_s^\zeta &= \zeta + \int_s^t g(r, Y_r^\zeta, Z_r^\zeta)dr - \int_s^t Z_r^\zeta dB_r. \end{aligned}$$

Their convex combination: $(Y_s, Z_s) := (\alpha Y_s^{\xi} + (1-\alpha)Y_s^{\zeta}, \alpha Z_s^{\xi} + (1-\alpha)Z_s^{\zeta})$, satisfies

$$Y_s = \alpha\xi + (1-\alpha)\zeta + \int_s^t [g(r, Y_r, Z_r) + \psi_r]dr - \int_s^t Z_r dB_r,$$

where we set $\psi_r = \alpha g(r, Y_r^{\xi}, Z_r^{\xi}) + (1-\alpha)g(r, Y_r^{\zeta}, Z_r^{\zeta}) - g(r, Y_r, Z_r)$.

But since the price generating function g is convex in (y, z), we have $\psi \geq 0$. It then follows from the comparison theorem that $Y_s \geq \mathbb{E}_{s,t}^g[\alpha\xi + (1-\alpha)\zeta]$, for each. We thus have **(ii)**.

(ii)⇒(i): Let $Y^{t,y,z}$ be the solution of SDE (5.3). For fixed $t \in [0, T)$ and (y, z), (y', z') in $R \times R^d$, we have

$$Y_s^{t,y,z} = \mathbb{E}_{s,t}^g[Y_t^{t,y,z}],\ Y_s^{t,y',z'} = \mathbb{E}_{s,t}^g[Y_t^{t,y',z'}].$$

We set $Y_s := \alpha Y_s^{t,y,z} + (1-\alpha)Y_s^{t,y',z'}$, $s \in [t_0, T]$. By (5.7),

$$\begin{aligned}\mathbb{E}_{s,t}^g[Y_t] &\leq \alpha\mathbb{E}_{s,t}^g[Y_s^{t,y,z}] + (1-\alpha)\mathbb{E}_{s,t}^g[Y_s^{t,y',z'}]\\ &= \alpha Y_s^{t,y,z} + (1-\alpha)Y_s^{t,y',z'}\\ &= Y_s.\end{aligned}$$

Thus the process Y is a g–supermartingale defined on $[t, T]$. It follows from the decomposition theorem, i.e., Theorem 4.13, that, there exists an increasing process $A \in D_{\mathcal{F}}^2(t, T)$ such that

$$Y_s = Y_t - \int_t^s g(r, Y_r, Z_r)dr - (A_s - A_t) + \int_t^s Z_r dB_s.$$

We compare this with

$$\begin{aligned}Y_s &= \alpha Y_s^{t,y,z} + (1-\alpha)Y_s^{t,y',z'}\\ &= \alpha y + (1-\alpha)y' - \int_t^s [\alpha g(r, Y_r^{t,y,z}, z) + (1-\alpha)g(r, Y_r^{t,y',z'}, z')]dr\\ &+ (\alpha z + (1-\alpha)z')(B_s - B_t),\end{aligned}$$

It follows that

$$\begin{gathered}Y_t = \alpha y + (1-\alpha)y',\ Z_r \equiv \alpha z + (1-\alpha)z',\\ g(r, Y_r, Z_r) \equiv g(r, \alpha Y_r^{t,y,z} + (1-\alpha)Y_r^{t,y',z'}, \alpha z + (1-\alpha)z').\end{gathered}$$

Thus we have

$$g(s, \alpha Y_s^{t,y,z} + (1-\alpha)Y_s^{t,y',z'}, \alpha z + (1-\alpha)z') \leq \alpha g(s, Y_s^{t,y,z}, z) + (1-\alpha)g(s, Y_s^{t,y',z'}, z').$$

We then can apply Lemma 5.3 to obtain (i). ■

Proposition 5.7 *The following two conditions are equivalent:*
(i) *The price generating function g is positively homogenous in $(y,z)\in R\times R^d$, i.e.,*

$$g(t,\lambda y,\lambda z)=\lambda g(t,y,z),\ a.e.,\ a,s,,$$

(ii) *The corresponding pricing mechanism $\mathbb{E}^g_{s,t}[\cdot]:L^2(\mathcal{F}_t)\longmapsto L^2(\mathcal{F}_s)$ is positively homogenous: for each $0\le s\le t$, i.e., $\mathbb{E}^g_{s,t}[\lambda\xi]=\lambda\mathbb{E}^g_{s,t}[\xi]$, for each $\lambda\ge 0$ and $\xi\in L^2(\mathcal{F}_t)$.*

Proof. (i)⇒(ii) is easy.
(ii)⇒(i): Let $Y^{t,y,z}$ be the solution of SDE (5.3). For fixed $t\in[0,T)$ and (y,z) in $R\times R^d$, we have $\lambda Y^{t,y,z}_s=\mathbb{E}^g_{s,t}[\lambda Y^{t,y,z}_t]$, $s\in[t,T]$. This implies that, there exists $Z^{t,y,z,\lambda}_\cdot\in L^2_{\mathcal{F}}(t,T;R^d)$, such that

$$\lambda Y^{t,y,z}_s=\lambda y-\int_t^s g(r,\lambda Y^{t,y,z}_r,Z^{t,y,z,\lambda}_r)dr+\int_t^s Z^{t,y,z,\lambda}_r dB_r,\ s\in[t,T].$$

Compare this with $\lambda Y^{t,y,z}_s=\lambda y-\int_t^s\lambda g(r,Y^{t,y,z}_r,z)dr+\int_t^s\lambda z dr$, it follows that $Z^{t,y,z,\lambda}_\cdot\equiv\lambda z$ and $\lambda g(r,Y^{t,y,z}_r,z)\equiv g(r,\lambda Y^{t,y,z}_r,Z^{t,y,z,\lambda}_r)$, $r\in[t,T]$. We then can apply Lemma 5.3 to obtain (i). ■

From the above two propositions we immediatly have

Corollary 5.8 *The following two conditions are equivalent:*
(i) *The price generating function g is subadditive: for each (y,z), $(y',z')\in R\times R^d$,*

$$g(\omega,t,y+y',z+z')\le g(\omega,t,y,z)+g(\omega,t,y',z'),\ dt\times dP,\ a.s.,$$

(ii) *The corresponding pricing mechanism $\mathbb{E}^g_{s,t}[\cdot]:L^2(\mathcal{F}_t)\longmapsto L^2(\mathcal{F}_s)$ is is subadditive: for each $0\le s\le t$,*

$$\mathbb{E}^g_{s,t}[\xi+\xi']\le\mathbb{E}^g_{s,t}[\xi]+\mathbb{E}^g_{s,t}[\xi'],\quad\forall\xi,\xi'\in L^2(\mathcal{F}_t).$$

Proposition 5.9 *The price generating function g is independent of y if and only if, the corresponding g–pricing mechanism is cash invariant, namely, for each $s\le t$*

$$\mathbb{E}^g_{s,t}[\xi+\eta]=\mathbb{E}^g_{s,t}[\xi]+\eta,\ a.s.,\forall\xi\in L^2(\mathcal{F}_t),\ \eta\in L^2(\mathcal{F}_s).$$

Proof. We first prove the "If" part. For each fixed $(y,z)\in R\times R^d$, we have $Y^{t,y,z}_s\equiv\mathbb{E}^g_{s,T}[Y^{t,y,z}_T]\equiv y+\mathbb{E}^g_{s,T}[Y^{t,y,z}_T-y]$. Let $\bar{Y}_s=\mathbb{E}^g_{s,T}[Y^{t,y,z}_T-y]$, $s\in[0,T]$ and $\bar{Z}$ be the corresponding part of Itô's integrand. By $\bar{Y}_r\equiv y+Y^{t,y,z}_r$ it follows that

$$\begin{aligned}y+Y_s&=y+Y^{t,y,z}_T+\int_s^T g(r,Y^{t,y,z}_r,z)-\int_s^T zdB_r\\&=(y+Y^{t,y,z}_T)+\int_s^T g(r,\bar{Y}_r,\bar{Z}_r)-\int_s^T\bar{Z}_r dB_r.\end{aligned}$$

Thus $\bar{Z}_r \equiv z$ and

$$g(r, Y_r^{t,y,z}, z) \equiv g(r, Y_r^{t,y,z} - y, \bar{Z}_r) \equiv g(r, Y_r^{t,y,z} - y, z).$$

We then can apply Lemma 5.3 to obtain that, for each $(y, z) \in R \times R^d$,

$$g(r, y, z) \equiv g(r, y - y, z) \equiv g(r, 0, z).$$

Namely, g is independent of y.
"Only if part": For each for each $s \le t$ and $\xi \in L^2(\mathcal{F}_t)$, $\eta \in L^2(\mathcal{F}_s)$, we have

$$Y_r := \mathbb{E}^g_{s,t}[\xi + \eta] = \xi + \eta + \int_r^t g(u, Z_u) du - \int_s^t Z_u dB_u,\ r \in [s, t].$$

Thus $\bar{Y}_r := Y_r - \eta$ is a g–solution on $[s, t]$ with terminal condition $\bar{Y}_t = \xi + \eta$. This implies

$$\mathbb{E}^g_{s,t}[\xi] + \eta = \bar{Y}_s = \mathbb{E}^g_{s,t}[\xi + \eta].$$

The proof is complete. ■

We consider the following self–financing condition:

$$\mathbb{E}^g_{s,t}[0] \equiv 0,\ \forall 0 \le s \le t. \tag{5.8}$$

Proposition 5.10 $\mathbb{E}^g[\cdot]$ *satisfies the self–financing condition if and only if its price generating function g satisfies*

$$g(t, 0, 0) = 0,\ a.e.,\ a.s.$$

Proof. The "if" part is obvious.
The "only if part": $Y_t := \mathbb{E}^g_{t,T}[0] \equiv 0$, implies

$$Y_t \equiv 0 \equiv 0 + \int_t^T g(s, 0, Z_s) ds - \int_t^T Z_s dB_s,\ t \in [0, T].$$

Thus $Z_s \equiv 0$ and then $g(s, 0, Z_s) = g(s, 0, 0) \equiv 0$. ■

Zero–interesting rate condition:

$$\mathbb{E}^g_{s,t}[\eta] = \eta,\ \forall 0 \le s \le t < \infty,\ \eta \in L^2(\mathcal{F}_s).$$

Proposition 5.11 $\mathbb{E}^g[\cdot]$ *satisfies the zero–interesting rate condition if and only if its price generating function g satisfies, for each $y \in R$,*

$$g(\cdot, y, 0) = 0.$$

Proof. For a fixed $y \in R$, we consider $Y_s := \mathbb{E}^g_{s,T}[y] \equiv y$. Let Z_s be the corresponding Itô's integrand

$$Y_t = y + \int_t^T g(s, Y_s, Z_s) - \int_t^T Z_s dB_s \equiv y.$$

But this is equivalent to

$$Y_t \equiv y,\ Z_s \equiv 0,\ g(s,y,0) \equiv 0.$$

■

For each $\bar{z}^{i_0}_\cdot \in L^2_{\mathcal{F}}(0,T)$

$$\mathbb{E}_{t,T}[\xi] + \int_0^t \bar{z}^{i_0}_s dB^{i_0}_s = \mathbb{E}_{t,T}[\xi + \int_t^T \bar{z}^{i_0}_s dB^{i_0}_s] \tag{5.9}$$

Proposition 5.12 *Condition 5.9 holds if and only if $g(s,y,z)$ does not depends on the i_0th component z^{i_0} of $z \in R^d$.*

Proof. The "if" part: Since process $Y_t := \mathbb{E}_{t,T}[\xi]$ solves the following BSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s,$$

we have

$$Y_t + \int_0^t \bar{z}^{i_0}_s dB^{i_0}_s = \xi + \int_0^T \bar{z}^{i_0}_s dB^{i_0}_s + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T \bar{Z}_s dB_s,$$

where

$$\bar{Z}_s = \left(Z^1_s, \cdots, Z^{i_0-1}_s, Z^{i_0}_s + \bar{z}^{i_0}_s, Z^{i_0+1}_s, \cdots, Z^d_s\right).$$

But since $g(s,y,z)$ does not depend the i_0th component of $z \in R^d$, we thus have $g(s, Y_s, Z_s) \equiv g(s, Y_s, \bar{Z}_s)$. Thus

$$Y_t + \int_0^t \bar{z}^{i_0}_s dB^{i_0}_s = \xi + \int_0^t \bar{z}^{i_0}_s dB^{i_0}_s + \int_t^T g(s, Y_s, \bar{Z}_s)ds - \int_t^T \bar{Z}_s dB_s.$$

This means that (5.9) holds.
The "only if" part: For each fixed (t,y,z), let $(Y^{t,y,z}_s)_{s\geq t}$ be the solution of (5.3). We have,

$$\mathbb{E}_{s,T}[Y^{t,y,z}_T] - z^{i_0} B^{i_0}_s = \mathbb{E}_{s,T}[Y^{t,y,z}_T - z^{i_0} B^{i_0}_T],\ s \in [t,T].$$

Since the process $Y_r := \mathbb{E}_{s,r}[Y^{t,y,z}_r - z^{i_0}_r B^{i_0}_r]$, $r \in [t,s]$, solves the BSDE

$$Y^{t,y,z}_s - z^{i_0} B^{i_0}_s = Y_s = Y^{t,y,z}_T + z^{i_0} B^{i_0}_T + \int_s^T g(r, Y_r, Z_r)ds - \int_s^T Z_r dB_r.$$

From which we deduce $Z_s = \bar{z} := \left(z^1, \cdots, z^{i_0-1}, 0, z^{i_0+1}, \cdots, z^d\right) = z$ and thus

$$g(r, Y_r, Z_r) = g(r, Y^{t,y,z}_r, \bar{z}) = g(r, Y^{t,y,z}_r, z),\ 0 \leq t \leq r \leq T.$$

It then follows from Lemma 5.3 that

$$g(t,y,\bar{z}) = g(t,y,z),\ t \geq 0, \text{ a.e., a.s.},$$

i.e., g does not depend the i_0th component of $z \in R^d$. ■

Proposition 5.13 *The following condition are equivalent:*
(i) *For each $0 \le s \le t$, and $X \in L^2(\mathcal{F}_t^s)$, the g–pricing mechanism $\mathbb{E}^g_{s,t}[X]$ is deterministic;*
(ii) *The corresponding pricing generating function g is a deterministic function of $(t, y, z) \in [0, T] \times R \times R^d$.*

The proof is similar as the others. We omit it.

6 Pricing accumulated contingent claim by a general $\mathbb{E}_{s,t}[\cdot]$

For a given $K \in D^2_{\mathcal{F}}(0, \infty)$, we will find the corresponding definition $\mathbb{E}_{s,t}[\cdot; K]$, for an abstract dynamic pricing mechanism $\mathbb{E}_{s,t}[\cdot]$ defined on $[0, \infty)$. To this end we first consider the case $K \in D^0_{\mathcal{F}}(0, \infty)$, the space of step processes defined by

$$D^0_{\mathcal{F}}(0, \infty) := \{K_t = \sum_{i=0}^{N-1} \xi_i 1_{[t_i, t_{i+1})}(t),\ \{t_i\}_{i=0}^T \in \pi^N_{[0,\infty)},\ \xi_i \in L^2(\mathcal{F}_{t_i})\}. \quad (6.1)$$

Now let $K_t = \sum_{i=0}^{N-1} \xi_i 1_{[t_i, t_{i+1})}(t)$ be fixed. We observe that, for each $T > 0$ and $X \in L^2(\mathcal{F}_T)$, (X, K) is an accumulated contingent claim with maturity T in a simple way that, at each time $t_i \le T$, the buyer of the contract (X, K) receives $K_{t_i} - K_{t_i -}$, and, in addition, she or he receives X at the maturity T.

We define, for each $0 \le i \le N-1$, $s, t \in [t_i, t_{i+1}]$, with $s \le t$ and $X \in L^2(\mathcal{F}_t)$,

$$\mathbb{E}^i_{s,t}[X; K] := \mathbb{E}_{s,t}[X + K_t - K_s].$$

Lemma 6.1 *For each $i = 0, 1, 2, \cdots, N-1$, $\mathbb{E}_{s,t}[\cdot; K]$, $t_i \le s \le t \le t_{i+1}$ is an $\mathcal{F}_t$–consistent pricing mechanism.*

Proof. It is easy to check that (A1), (A2) and (A3) holds. We now prove (A4), i.e., for each $t_i \le s \le t \le t_{i+1}$ and $X \in L^2(\mathcal{F}_t)$,

$$1_A \mathbb{E}^i_{s,t}[X; K] = 1_A \mathbb{E}^i_{s,t}[1_A X; K],\ \forall A \in \mathcal{F}_s. \quad (6.2)$$

We have

$$\begin{aligned} 1_A \mathbb{E}^i_{s,t}[X; K.] &= 1_A \mathbb{E}_{s,t}[X + K_t - K_s] \\ &= 1_A \mathbb{E}_{s,t}[1_A(1_A X + K_t - K_s)] \\ &= 1_A \mathbb{E}_{s,t}[1_A X + K_t - K_s] \\ &= 1_A \mathbb{E}^i_{s,t}[1_A X; K.]. \end{aligned}$$

Thus (A4) holds. ■

By Proposition 2.8, there exists a unique $\mathcal{F}_t$–consistent pricing mechanism $\mathbb{E}[\cdot; K]$, that coincides with $\mathbb{E}^i[\cdot; K]$ for each interval $[t_i, t_{i+1}]$.

Definition 6.2 *We denote this unique $\mathcal{F}_t$–consistent pricing mechanism that coincides with $\mathbb{E}^i[\cdot; K]$ by $\mathbb{E}_{s,t}[\cdot; K]$:*

$$\mathbb{E}_{s,t}[X; K] : X \in L^2(\mathcal{F}_t) \to L^2(\mathcal{F}_s).$$

Remark 6.3 *It is easy to check that, for each accumulated contingent claim (X, K) with maturity t and $K \in D^0_{\mathcal{F}}(0, t)$, the only consistent price of (X, K) of the pricing mechanism $\mathbb{E}$ is $\mathbb{E}_{s,t}[X; K]_{0\le s\le t}$.*

Proposition 6.4 *We assume that $\mathbb{E}_{s,t}[\cdot]_{0\le s\le t<\infty}$ is a given pricing mechanism satisfying (A1)–(A5) and (A4$_0$). Then for each $K \in D^2_{\mathcal{F}}(0, \infty)$ there exists a pricing mechanism $\mathbb{E}_{s,t}[\cdot; K]_{0\le s\le t<\infty}$ which is dominated by $\mathbb{E}^{g_\mu}$ in the following sense, for each K, $K' \in D^2_{\mathcal{F}}(0, \infty)$ and for each $0 \le s \le t$, X, $X' \in L^2(\mathcal{F}_t)$, we have*

$$\begin{aligned} \mathbb{E}^{-g_\mu}_{s,t}[X - X'; (K - K').] &\le \mathbb{E}_{s,t}[X; K.] - \mathbb{E}_{s,t}[X'; K'.] \\ &\le \mathbb{E}^{g_\mu}_{s,t}[X - X'; (K - K').],\ a.s. \end{aligned} \tag{6.3}$$

and

$$\mathbb{E}^{-g_\mu}_{s,t}[0; K.] \le \mathbb{E}_{s,t}[0; K.] \le \mathbb{E}^{g_\mu}_{s,t}[0; K.] \tag{6.4}$$

Moreover, for each $K_t = \sum_{i=0}^{N-1} \xi_i I_{[t_i,t_{i+1})}(t)$, we have

$$\mathbb{E}_{s,t}[X; K.] = \mathbb{E}_{s,t}[X + K_t - K_s],\ \forall [s, t] \in [t_i, t_{i+1}]$$

Such pricing mechanism is uniquely defined. Furthermore, under the pricing mechanism $\mathbb{E}$ the price process of the accumulated contingent claim (X, K) with maturity t is $\mathbb{E}_{s,t}[X; K.]$, $s \le t$.

The prove of this proposition can be found in [43].

7 $\mathbb{E}[\cdot; K]$–martingales

Hereinafter, $\mathbb{E}_{s,t}[\cdot]$ will be a fixed $\mathcal{F}_t$–consistent pricing mechanism satisfying (A1)–(A5) and (A4$_0$). Similar to $\mathbb{E}^g_{s,t}[\cdot]$-pricing mechanism, we introduce the notion of $\mathbb{E}[\cdot; K]$–martingale:

Definition 7.1 *Let $K \in D^2_{\mathcal{F}}(0, T)$ be given. A process $Y \in L^2_{\mathcal{F}}(t_0, t_1)$ satisfying $E[\operatorname{ess\,sup}_{s\in[t_1,t_1]} |Y_s|^2] < \infty$, is said to be an $\mathbb{E}[\cdot; K]$–martingales (resp. $\mathbb{E}[\cdot; K]$–supermartingale, $\mathbb{E}[\cdot; K]$–submartingale) on $[t_0, t_1]$ if for each $t_0 \le s \le t \le t_1$, we have*

$$\mathbb{E}_{s,t}[Y_t; K] = Y_s,\ (resp.\ \le Y_s,\ \ge Y_s),\ a.s. \tag{7.1}$$

We then can apply $\mathbb{E}^g$–supermartingale decomposition theorem, i.e., Proposition 4.13, to get the following result.

Proposition 7.2 *We assume (A1)–(A5) and (A4$_0$). Let $K \in D^2_{\mathcal{F}}(0,T)$ be given. For fixed $t \in [0,T]$ and $X \in L^2(\mathcal{F}_t)$, the process $Y_s^{t,X,K} := \mathbb{E}_{s,t}[X;K]$, $s \in [0,t]$, has the following expression: there exist processes $(g^{t,X,K}_\cdot, z^{t,X,K}_\cdot) \in L^2_{\mathcal{F}}(0,t;R\times R^d)$ such that*

$$Y_s^{t,X,K} = X + K_t - K_s + \int_s^t g_r^{t,X,K} dr - \int_s^t z_r^{t,X,K} dB_r, \quad s \in [0,t], \tag{7.2}$$

such that

$$|g_s^{t,X,K}| \le \mu(|Y_s^{t,X,K}| + |z_s^{t,X,K}|), \ \forall s \in [0,t]. \tag{7.3}$$

Moreover let $Y_s^{t,X',K'} := \mathbb{E}_{s,t_1}[X';K']$, $s \in [0,t]$, for some other $K' \in D^2_{\mathcal{F}}(0,T)$, $X' \in L^2(\mathcal{F}_t)$ and let $(g^{t,X',K'}_\cdot, z^{t,X',K'}_\cdot)$ be the corresponding expression in (7.2), then we have

$$|g_s^{t,X,K} - g_s^{t,X',K'}| \le \mu(|Y_s^{t,X,K} - Y_s^{t,X',K'}| + |z_s^{t,X,K} - z_s^{t,X',K'}|), \ \forall s \in [0,t]. \tag{7.4}$$

Proof. Since $(Y_s^{t,X,K})_{s\in[0,t]}$, is an $\mathbb{E}^{g_\mu}[\cdot;K]$–submartingale and $\mathbb{E}^{-g_\mu}[\cdot;K]$–super–martingale, by Proposition 4.13 and Corollary 4.14, there exists an increasing process $A^+_\cdot \in D^2_{\mathcal{F}}(0,t)$ and $A^-_\cdot \in D^2_{\mathcal{F}}(0,t)$ with $A^+_0 = A^-_0 = 0$, such that

$$Y_s^{t,X,K} = \mathbb{E}^{g_\mu}_{s,t}[X;(K-A^+)_\cdot] = \mathbb{E}^{g_\mu}_{s,t}[X;(K+A^-)_\cdot], \ s \in [0,t]. \tag{7.5}$$

According to the notion of $\mathbb{E}^g$ defined in (4.5), $Y_s^{t,X,K}$ is the solution of the following BSDE on $[0,t]$:

$$\begin{aligned} Y_s^{t,X,K} &= X + (K-A^+)_t - (K-A^+)_s \\ &\quad + \int_s^t \mu(|Y_r^{t,X,K}| + |Z_r^+|)dr - \int_s^t Z_r^+ dB_r \end{aligned} \tag{7.6}$$

and

$$\begin{aligned} Y_s^{t,X,K} &= X + (K+A^-)_t - (K+A^-)_s \\ &\quad - \int_s^t \mu(|Y_r^{t,X,K}| + |Z_r^-|)dr - \int_s^t Z_r^- dB_r. \end{aligned} \tag{7.7}$$

It then follows that $Z_s^{t,X,K} := Z_s^+ \equiv Z_s^-$, $s \in [0,t]$ and thus

$$-dA_s^+ + \mu(|Y_s^{t,X,K}| + |Z_s^{t,X,K}|)ds \equiv dA_s^- - \mu(|Y_s^{t,X,K}| + |Z_s^{t,X,K}|)ds,$$

or

$$dA_s^- + dA_s^+ \equiv 2\mu(|Y_s^{t,X,K}| + |Z_s^{t,X,K}|)ds, \ s \in [0,t] \tag{7.8}$$

Thus dA^+ and dA^- are absolutely continuous with respect to ds. We denote $a_s^+ ds = dA_s^+$ and $a_s^- ds = dA_s^-$. It is clear that

$$\begin{aligned} &0 \le a_s^+ \le 2\mu(|Y_s^{t,X,K}| + |Z_s^{t,X,K}|), \\ &0 \le a_s^- \le 2\mu(|Y_s^{t,X,K}| + |Z_s^{t,X,K}|), \ dP \times dt\text{–a.e.} \end{aligned}$$

We then can rewrite (7.6) as

$$Y_s^{t,X,K} = X + K_t - K_s + \int_s^t [-a_r^+ + \mu(|Y_r^{t,X,K}| + |Z_r^+|)]dr - \int_s^t Z_r^+ dB_r. \quad (7.9)$$

Thus, by setting $g_r^{t,X,K} := -a_r^+ + \mu(|Y_r^{t,X,K}| + |Z_r^+|)$, we have the expression (7.2) as well as the estimate (7.3). ■

It remains to prove (7.4). By (A5) of Proposition 4.13 $\hat{Y}_s = Y_s^{t,X,K} - Y_s^{t,X',K'}$ is an $\mathbb{E}^{g_\mu}[\cdot; K-K']$–submartingale and an $\mathbb{E}^{-g_\mu}[\cdot; K-K']$–supermartingale on $[0,t]$. Thus we can repeat the above procedure to prove that there exist processes $(\hat{g}_\cdot, \hat{Z}_\cdot) \in L^2_{\mathcal{F}}(0,t; R\times R^d)$ such that

$$\hat{Y}_s = X - X' + (K-K')_t - (K-K')_s + \int_s^t \hat{g}_r dr - \int_s^t \hat{Z}_r dB_r, \quad s \in [0,t], \quad (7.10)$$

such that

$$|\hat{g}_s| \le \mu(|\hat{Y}_s| + |\hat{Z}_s|), \ \forall s \in [0,t]. \quad (7.11)$$

But by (7.2) and $\hat{Y}_s \equiv Y_s^{t,X,K} - Y_s^{t,X',K'}$, we immediately have

$$\hat{g}_s \equiv g_s^{t,X,K} - g_s^{t,X',K'}, \ \hat{Z}_s \equiv z_s^{t,X,K} - z_s^{t,X',K'}. \quad (7.12)$$

This with (7.11) yields (7.4). The proof is complete.

Corollary 7.3 *Let K^1 and $K^2 \in D^2_{\mathcal{F}}(0,T)$ and $X^1 \in L^2(\mathcal{F}_{t_1})$, $X^2 \in L^2(\mathcal{F}_{t_2})$ be given for some fixed $0 \le t_1 \le t_2 \le T$ and let $(g_s^{t_i,X^i,K^i}, Z_s^{t_i,X^i,K^i})_{s\in[0,t_i]}$, $i=1,2$, be the pair in (7.2) for $Y_s^{t_i,X^i,K^i} = \mathbb{E}_{s,t_i}[X^i; (K^i)_\cdot]$, $i=1,2$, respectively. Then we have*

$$|g_s^{t,X^1,K^1} - g_s^{t,X^2,K^2}| \le \mu(|Y_s^{t_1,X^1,K^2} - Y_s^{t_2,X^2,K^2}| + |z_s^{t_1,X^1,K^1} - z_s^{t_2,X^2,K^2}|), \ \forall s \in [0,t_1]. \quad (7.13)$$

Proof. With the observation

$$Y_s^{t_2,X^2,K^2} = \mathbb{E}_{s,t_1}[Y_{t_1}^{t_2,X^2,K^2}; (K^2)_\cdot], \ s \in [0,t_1],$$

it is an immediate consequence of Proposition 7.2. ■

Corollary 7.4 *For each $t \in [0,T]$ and $X \in L^2(\mathcal{F}_t)$, $K \in D^2_{\mathcal{F}}(0,T)$, the process $(\mathbb{E}_{s,t}[X; K_\cdot])_{s\in[0,t]}$ is also in $D^2_{\mathcal{F}}(0,t)$. If moreover, $K \in S^2_{\mathcal{F}}(0,T)$ (resp. Itô's process), then $(\mathbb{E}_{s,t}[X; K_\cdot])_{s\in[0,t]}$ is also in $S^2_{\mathcal{F}}(0,t)$ (resp. Itô's process).*

8 $\mathbb{E}$–supermartingale decomposition theorem: intrinsic formulation

Our objective of this section is to give the following $\mathbb{E}$–supermartingale decomposition theorem of Doob–Meyer's type. Since $(\mathbb{E}_{s,t}[\cdot])_{s\le t}$ is abstract and nonlinear, it is necessary to introduce the intrinsic form (8.1). This theorem plays an important role in the proof of the main theorem of this paper. It can be also considered as a generalization of Proposition 4.13. This is a very profond theorem, the proof can be found in [43, Peng 2005].

Theorem 8.1 *We assume (A1)–(A5) as well as (A4$_0$). Let $Y \in S^2_{\mathcal{F}}(0,T)$ be an $\mathbb{E}[\cdot]$–supermartingale. Then there exists an increasing process $A \in S^2_{\mathcal{F}}(0,T)$ with $A_0 = 0$, such that Y is an $\mathbb{E}[\cdot; A]$–martingale, i.e.,*

$$Y_t = \mathbb{E}_{t,T}[Y_T; A_\cdot], \ t \in [0,T]. \tag{8.1}$$

Remark 8.2 *This theorem has an interesting interpretation: the fact that $Y \in S^2_{\mathcal{F}}(0,T)$ is an $\mathbb{E}[\cdot]$–supermartingale means that if Y is always undervalued by the pricing mechanism $\mathbb{E}$, i.e., $\mathbb{E}_{t,T}[Y_T] \leq Y_t$, then there exists an increasing process A such that Y_t is just the $\mathbb{E}$ price of the accumulated contingent claim (Y_T, A) at maturity T.*

Remark 8.3 *In the case where $(\mathbb{E}_{s,t}[\cdot])_{0\leq s\leq t\leq T}$ is a system of linear mappings, (8.1) becomes*

$$Y_t + A_t = \mathbb{E}_{t,T}[Y_T + A_T], \ t \in [0,T],$$

i.e., as in classical situation, $Y + A$ is an $\mathbb{E}[\cdot]$–martingale. But, the intrinsic formulation that can be applied to nonlinear situation is that Y is an $\mathbb{E}[\cdot; A]$–martingale.

9 Appendix

9.1 Proof of Theorem 3.1

For each fixed $(t, y, z) \in [0,T] \times R \times R^d$, we consider the solution $Y^{t,y,z} \in S^2_{\mathcal{F}}(0,T)$ of a Itô's equation on $[t,T]$:

$$dY^{t,y,z}_s = -g_\mu(Y^{t,y,z}_s, z)ds + zdB_s, \ \ s \in (t,T], \tag{9.1}$$

$$Y^{t,y,z}_t = y. \tag{9.2}$$

It is easy to check that $Y^{t,y,z}$ is an $\mathbb{E}^{g_\mu}[\cdot]$–martingale, i.e., it is a price process of the pricing mechanism $\mathbb{E}^{g_\mu}[\cdot]$ on $[t,T]$. Since the pricing mechanism $\mathbb{E}[\cdot]$ is dominated by $\mathbb{E}^{g_\mu}[\cdot]$, from (3.1) $Y^{t,y,z}$ is also an $\mathbb{E}[\cdot]$–supermartingale. By Decomposition Theorem 8.1, there exists an increasing process $A^{t,y,z} \in S^2_{\mathcal{F}}(0,T)$ with $A^{t,y,z}_0 = 0$, such that

$$Y^{t,y,z}_s = \mathbb{E}_{s,T}[Y^{t,y,z}_T; A^{t,y,z}_\cdot]. \tag{9.3}$$

i.e., $Y^{t,y,z}_\cdot$ is just the pricing process produced by $\mathbb{E}[\cdot]$ of the accumulated contingent claim $(Y^{t,y,z}_T; A^{t,y,z}_\cdot)$ with maturity T. By Proposition 7.2 and Corollary 7.3, there exists $(\ g^{t,y,z}, Z^{t,y,z}) \in L^2_{\mathcal{F}}(0,T)$ such that

$$-\,dY^{t,y,z}_s = dA^{t,y,z}_s + g^{t,y,z}_s ds - Z^{t,y,z}_s dB_s, \ s \in [t,T], \tag{9.4}$$

and such that, for each different (t,y,z), $(t',y',z') \in [0,T] \times R \times R^d$

$$|g^{t,y,z}_s - g^{t',y',z'}_s| \leq \mu|Y^{t,y,z}_s - Y^{t',y',z'}_s| + \mu|Z^{t,y,z}_s - Z^{t',y',z'}_s|, \ s \in [t \vee t', T], \tag{9.5}$$

and

$$|g_s^{t,y,z}| \le \mu |Y_s^{t,y,z}| + \mu |Z_s^{t,y,z}|,\ s \in [t,T],\ ds \times dP\text{–a.e.} \tag{9.6}$$

Now for each $X \in L^2(\mathcal{F}_{t'})$, we set

$$\bar{Y}_s^{t',X} := \mathbb{E}_{s,t'}[X] = \mathbb{E}_{s,t'}[X;0]. \tag{9.7}$$

We use once more Proposition 7.2 and Corollary 7.3: there exists $(\bar{g}^{t',X}, \bar{Z}^{t',X}) \in L^2_{\mathcal{F}}(0,t')$ such that, for $s \in [0,t']$,

$$-d\bar{Y}_s^{t',X} = \bar{g}_s^{t',X} ds - \bar{Z}_s^{t',X} dB_s,\ \bar{Y}_{t'} = X, \tag{9.8}$$

such that

$$|g_s^{t,y,z} - \bar{g}_s^{t',X}| \le \mu |Y_s^{t,y,z} - \bar{Y}_s^{t',X}| + \mu |Z_s^{t,y,z} - \bar{Z}_s^{t',X}|,\ s \in [t,t'],\ ds \times dP\text{–a.e.} \tag{9.9}$$

and, for X, $X \in L^2(\mathcal{F}_{t'})$,

$$|\bar{g}_s^{t',X} - \bar{g}_s^{t',X'}| \le \mu |\bar{Y}_s^{t',X} - \bar{Y}_s^{t',X'}| + \mu |\bar{Z}_s^{t',X} - \bar{Z}_s^{t',X'}|,\ s \in [0,t'],\ ds \times dP\text{–a.e..}$$

On the other hand, comparing to (9.1) and (9.4), we have

$$Z_s^{t,y,z} \equiv 1_{[t,T]}(s) z.$$

Thus (9.5), (9.6) and (9.9) become, respectively,

$$|g_s^{t,y,z} - g_s^{t',y',z'}| \le \mu |Y_s^{t,y,z} - Y_s^{t',y',z'}| + \mu |z - z'|,\ s \in [t \vee t', T],\ ds \times dP\text{–a.e.,} \tag{9.10}$$

$$|g_s^{t,y,z}| \le \mu |Y_s^{t,y,z}| + \mu |z|, \tag{9.11}$$

and

$$|g_s^{t,y,z} - \bar{g}_s^{t',X}| \le \mu |Y_s^{t,y,z} - \bar{Y}_s^{t',X}| + \mu |z - \bar{Z}_s^{t',X}|,\ s \in [t,t'],\ ds \times dP\text{–a.e.} \tag{9.12}$$

Now, for each $n = 1, 2, 3, \cdots$, we set $t_i^n = i 2^{-n} T$, $i = 0, 1, 2, \cdots, 2^n$, and define

$$g^n(s,y,z) := \sum_{i=0}^{2^n-1} g_s^{t_i^n,y,z} 1_{[t_i^n, t_{i+1}^n)}(s),\ s \in [0,T],\ (y,z) \in R \times R^d. \tag{9.13}$$

It is clear that g^n is an $\mathcal{F}_t$–adapted process.

Lemma 9.1 *For each fixed $(y,z) \in R \times R^d$ and $T > 0$, $\{g^n(\cdot,y,z)\}_{n=1}^\infty$ is a Cauchy sequence in $L^2_{\mathcal{F}}(0,T)$.*

Proof. Let $0 < m < n$ be two integers. For each $s \in [0,T)$, there are some integers $i \le 2^m - 1$ and $j \le 2^n - 1$ with $t_i^m \le t_j^m$, such that $s \in [t_i^m, t_{i+1}^m) \cap [t_j^n, t_{j+1}^n)$. We have, by (9.10)

$$\begin{aligned}
|g^m(s,y,z) - g^n(s,y,z)| &= |g_s^{t_i^m,y,z} - g_s^{t_j^n,y,z}| \\
&\le \mu |Y_s^{t_i^m,y,z} - Y_s^{t_j^n,y,z}| \\
&\le \mu |Y_s^{t_i^m,y,z} - y| + \mu |Y_s^{t_j^n,y,z} - y|.
\end{aligned}$$

By (5.4) of Lemma 5.1 given later,

$$\begin{aligned}&E[|g^m(s,y,z)-g^n(s,y,z)|^2]\\&\leq 2\mu^2C(|y|^2+|z|^2+1)(2^{-m}+2^{-n})T.\end{aligned}$$

Thus

$$\begin{aligned}\sup_{s\in[0,T)} E[|g^m(s,y,z)-g^n(s,y,z)|^2] &\leq 2\mu^2 E[|Y_s^{t_i^m,y,z}-y|^2+|Y_s^{t_j^n,y,z}-y|^2] \\ &\leq 2\mu^2C(|y|^2+|z|^2+1)(2^{-m}+2^{-n})T.\end{aligned} \tag{9.14}$$

Thus $\{g^n(\cdot,y,z)\}_{n=1}^\infty$ is a Cauchy sequence in $L^2_{\mathcal{F}}(0,T)$. ■

We can give

Definition 9.2 *For each $(y,z)\in R\times R^d$, we denote $g(\cdot,y,z)\in L^2_{\mathcal{F}}(0,T)$, the Cauchy limit of $\{g^n(\cdot,y,z)\}_{n=1}^\infty$ in $L^2_{\mathcal{F}}(0,T)$.*

We will prove that the pricing mechanism $\mathbb{E}_{s,t}[\cdot]$ is just the g–pricing mechanism with g obtained in the above definition as its generating function, and thus our main result Theorem 3.1 hold true. We still need to investigate some important properties of g. We have the following estimates for the function g.

Lemma 9.3 *The limit $g:\Omega\times[0,T]\times R\times R^d\to R^d$ satisfies the following properties:*

$$\left\{\begin{array}{ll} (i) & g(\cdot,y,z)\in L^2_{\mathcal{F}}(0,T), \text{ for each } (y,z)\in R\times R^d;\\ (ii) & |g(s,y,z)-g(s,y',z')|\leq \mu(|y-y'|+|z-z'|),\ \forall y,y'\in R,\ z,z'\in R^d;\\ (iii) & g(s,0,0)\equiv 0;\\ (iv) & |g(s,y,z)-\bar g^{t,X}|\leq \mu|y-\bar Y_s^{t,X}|+\mu|z-\bar Z_s^{t,X}|, \forall s\in[0,t],\ X\in L^2(\mathcal{F}_t).\end{array}\right. \tag{9.15}$$

where $(\bar Y^{t,X},\bar Z^{t,X})$ is the process defined in (9.7) and (9.8).

Proof. (i) is clear. To prove (ii), we choose $t_i^n=i2^{-n}T$, $i=0,1,2,\cdots,2^n$ as in (9.13). For each $s\in[0,T)$. We have, once more by (9.10),

$$\begin{aligned}|g^n(s,y,z)-g^n(s,y',z')| &= \sum_{j=0}^{2^n-1} 1_{[t_j^n,t_{j+1}^n)}(s)|g_s^{t_j^n,y,z}-g_s^{t_j^n,y,z}| \\ &\leq \mu\sum_{j=0}^{2^n-1} 1_{[t_j^n,t_{j+1}^n)}(s)(|Y_s^{t_j^n,y,z}-Y_s^{t_j^n,y,z}|+|z-z'|)\\ &\leq \mu\sum_{j=0}^{2^n-1} 1_{[t_j^n,t_{j+1}^n)}(s)(|Y_s^{t_j^n,y,z}-y|+|Y_s^{t_j^n,y,z}-y'|)\\ &+\mu(|y-y'|+|z-z'|)\end{aligned} \tag{9.16}$$

The first term $I^n(s)$ of the right hand is dominated by, using (5.4),

$$E[|I^n(s)|^2] \leq 2\mu^2 \sum_{i=0}^{2^n-1} 1_{[t_j^n, t_{j+1}^n)}(s) E[|Y_s^{t_j^n,y,z} - y|^2 + |Y_s^{t_j^n,y',z'} - y'|^2]$$
$$\leq 2\mu^2 \sum_{i=0}^{2^n-1} 1_{[t_j^n, t_{j+1}^n)}(s) C(|y|^2 + |z|^2 + |y'|^2 + |z'|^2 + 2)2^{-n}T.$$

Thus $I^n(\cdot) \to 0$ in $L^2_{\mathcal{F}}(0,T)$ as $n \to \infty$. (ii) is obtained by passing to the limit in both sides of (9.16). (iii) is proved similarly by using (9.11) and (5.4).
To prove (iv), We apply (9.12), ■

$$|g^n(s,y,z) - \bar{g}_s^{t,X}| = \sum_{i=0}^{2^n-1} 1_{[t_j^n, t_{j+1}^n)}(s) |g_s^{t_j^n,y,z} - \bar{g}_s^{t,X}|$$
$$\leq \sum_{i=0}^{2^n-1} 1_{[t_j^n, t_{j+1}^n)}(s) [\mu|Y_s^{t_j^n,y,z} - \bar{Y}_s^{t,X}| + \mu|z - \bar{Z}_s^{t,X}|]$$
$$\leq \mu \sum_{i=0}^{2^n-1} 1_{[t_j^n, t_{j+1}^n)}(s) |Y_s^{t_j^n,y,z} - y| + \mu|y - \bar{Y}_s^{t,X}| + \mu|z - \bar{Z}_s^{t,X}|.$$

Then we pass to the limit on both sides.

Finally, We give

Proof of Theorem 3.1. For each fixed t and $X \in L^2(\mathcal{F}_t)$, we denote by $\bar{Y}_s^{t,X} := \mathbb{E}_{s,t}[X]$, the $\mathbb{E}$-price on $s \in [0,t]$ with the contingent $\bar{Y}_t^{t,X} = X$ at the maturity X. By Proposition 7.2 and Corollary 7.3, this price process can has the form

$$\bar{Y}_s^{t,X} = X + \int_s^t \bar{g}_r^{t,X} dr - \int_s^t \bar{Z}_r^{t,X} dB_r, \; s \in [0,t].$$

On the other hand, let $Y_s^{t,X} = \mathbb{E}_{s,t}^g[X]$, the price process of the contingent claim X generated by g. It solves the BSDE

$$Y_s^{t,X} = X + \int_s^t g(r, Y_r^{t,X}, Z_r^{t,X}) dr - \int_s^t Z_r^{t,X} dB_r, \; s \in [0,t].$$

By Lemma 9.3–(i) and (ii), this BSDE is well–posed. We then apply Itô's formula to $|\bar{Y}^{t,X} - Y|^2$ on the pricing interval $[0,t]$, take expectation. Exactly as the classical proof of the uniqueness of BSDE, we have, using (iv) of Lemma

9.3.

$$
\begin{aligned}
E|\bar{Y}_s^{t,X} - Y_s|^2 + E\int_s^t |\bar{Z}_r^{t,X} - Z_r|^2 dEr \\
&= 2E\int_s^t (\bar{Y}_r^{t,X} - Y_r)(\bar{g}_r^{t,X} - g(r, Y_r, Z_r))dr \\
&\le 2E\int_s^t (|\bar{Y}_r^{t,X} - Y_r| \cdot |\bar{g}_r^{t,X} - g(r, Y_r, Z_r)|)dr \\
&\le 2E\int_s^t |\bar{Y}_r^{t,X} - Y_r| \cdot \mu(|\bar{Y}_r^{t,X} - Y_r| + |\bar{Z}_r^{t,X} - Z_r|)dr \\
&\le E\int_s^t 2(\mu + \mu^2)|\bar{Y}_r^{t,X} - Y_r|^2 + \frac{1}{2}|\bar{Z}_r^{t,X} - Z_r|^2)dr.
\end{aligned}
$$

It then follows by using Gronwall's inequality that

$$\bar{Y}_s^{t,X} \equiv Y_s = \mathbb{E}_{s,t}^g[X],\ \forall s \in [0, t].$$

We thus have the desired result. The proof is complete. □

9.2 Testing condition of domination (A5)

with computational realization by CHEN Lifeng and SUN Peng

With Chen and Sun of our research group, we have applied our main result 3.1 to test if a specific pricing mechanism is a g–expectation, or g–pricing mechanism. We need to verify by testing if the crucial assumption (A5), i.e., the domination inequality (3.1) holds true. We have tested by using the data of prices of different options given by the price mechanism.

We first test the CME (Chicago Mercantile Exchange)'s market price mechanism of the options with S&P500 futures as the underlying asset. The data of the call and put priceses, from year 2000 to 2003, and the corresponding S&P500 future's prices is obtained from the parameter files of SPAN (Standard Portfolio Analysis of Risk) system downloaded from CME's ftp site.

We denote by $X_T^i = (S_T - k_i)^+$ (resp. $Y_T^i = (S_T - k_i)^-$, the market price of the call (resp. put) option with muturity T and strike price k_i. We denote their market price at time $t < T$ by $\mathbb{E}_{t,T}^m[X_T^i]$ and $\mathbb{E}_{t,T}^m[Y_T^i]$, respectively. The inequalities we need to put to the test are (3.1) in the following different conbinations, with different (t, T) and different strike prices

$$
\begin{cases}
\text{Call–Call:} & \mathbb{E}_{t,T}^m[X_T^i] - \mathbb{E}_{t,T}^m[X_T^j] \le \mathbb{E}_{t,T}^{g_\mu}[X_T^i - X_T^j] \\
\text{Put–Put:} & \mathbb{E}_{t,T}^m[Y_T^i] - \mathbb{E}_{t,T}^m[Y_T^j] \le \mathbb{E}_{t,T}^{g_\mu}[Y_T^i - Y_T^j] \\
\text{Call–Put:} & \mathbb{E}_{t,T}^m[X_T^i] - \mathbb{E}_{t,T}^m[Y_T^j] \le \mathbb{E}_{t,T}^{g_\mu}[X_T^i - Y_T^j] \\
\text{Put–Call:} & \mathbb{E}_{t,T}^m[Y_T^i] - \mathbb{E}_{t,T}^m[X_T^j] \le \mathbb{E}_{t,T}^{g_\mu}[Y_T^i - X_T^j]
\end{cases}
\tag{9.17}
$$

In the above inequalities the left hand are market data taken from CME parameter files. The right hand are the corresponding g_μ–expectations. We have calculated all these values by using standard binomial tree algorithm of BSDE. Here use the algorithms in Peng and Xu [2005] to solve the following 1-dimensional BSDE:

$$y_t = \xi + \int_t^T \mu(|y_s| + |z_s|)ds - \int_t^T z_s dB_s \tag{9.18}$$
$$y_T = X_T^i - X_T^j \ \text{ (resp. } Y_T^i - Y_T^j,\ X_T^i - Y_T^j \text{ and } Y_T^i - X_T^j).$$

5 parameter files from year 2000 to 2003 have been put in the test. We list the number of tested inequalities (9.17) corresponding to each CME parameter file:

CME parameter file name	year	number of tested inequalities
cme0105s.par	2000	54584
cme0105s.par	2001	62424
cme0104s.par	2002	35830
cme0103s.par	2003	28162
cme0701s.par	2003	61438
total number tested		242438

This means that BSDE (9.18) have been caculated 242438 times (with CPU P4 Xeron 2.8G). A surpricingly positive result was obtained: among the totally 242438 tested inqualities, only 5 are against the criteria (9.17). Moreover, those 5 counterexamples are singular situation since they themself all violate Axiomatic monotonicity condition (A1). 5 cases are all from cme0701s.par, 2003, Put–Put. They are all the singular cases of form

$$\mathbb{E}^m_{t,T}[(S_T - k_i)^-] > \mathbb{E}^m_{t,T}[(S_T - k_j)^-], \quad \text{for } k_i > k_j.$$

More specific results of the test will be given in our forthcoming paper.

Another feature of our test is, usualy, the bigger $T-t$ is, the $\mathbb{E}^{g_\mu}_{t,T}[X_T^i - X_T^j] - (\mathbb{E}^m_{t,T}[X_T^i] - \mathbb{E}^m_{t,T}[X_T^j])$. We present 4 features for a relatively smaller $T-t$ to show the tested result.

Test of inequalities: CME file: cme0105s.par, 2001, t: Jan. 05, T: Jan. 17 for S&P500 01-03 future with $S_t = \$1413.5$

1. Call–Call: $\mathbb{E}^{g_\mu}_{t,T}[X_T^i - X_T^j] - (\mathbb{E}^m_{t,T}[X_T^i] - \mathbb{E}^m_{t,T}[X_T^j]) > 0$
2. Put–Put: $\mathbb{E}^{g_\mu}_{t,T}[Y_T^i - Y_T^j] - (\mathbb{E}^m_{t,T}[Y_T^i] - \mathbb{E}^m_{t,T}[Y_T^j]) > 0$
3. Call–Put: $\mathbb{E}^{g_\mu}_{t,T}[X_T^i - Y_T^j] - (\mathbb{E}^m_{t,T}[X_T^i] - \mathbb{E}^m_{t,T}[Y_T^j]) > 0$
4. Put–Call $\mathbb{E}^{g_\mu}_{t,T}[Y_T^i - X_T^j] - (\mathbb{E}^m_{t,T}[Y_T^i] - \mathbb{E}^m_{t,T}[X_T^j]) > 0$

Remark: the computations was realized by CHEN Lifeng and SUN Peng

References

[1] Anderson, E., L. P. Hansen and T.J. Sargent (2000) Robustness, detection and the price of risk, *Mimeo, Stanford University.*

[2] Artzner, P., Delbaen, F., Eber, J.M. & Heath, D. Coherent measures of risk, Math. Finance, **9**, 203–228, 1999.

[3] Barrieu, P. and El Karoui, N. (2004) Pricing, Hedging and Optimally Designing Derivatives via Minimization of Risk Measures, Preprint, to appear in Contemporary Mathematics.

[4] Barrieu, P. and El Karoui, N. (2005) Pricing, Hedging and Optimally Designing Derivatives via Minimization of Risk Measures, Preprint.

[5] J.-M. Bismut (1973) Conjugate convex functions in optimal stochastic control. J.Math. Anal. Appl., **44**, 384–404.

[6] J.-M. Bismut, (1978) An introductory approach to duality in optimal stochastic control. SIAM Rev., **20**(1), 62–78.

[7] P. Briand, F. Coquet, Y. Hu, J. Mémin and S. Peng, (2000) A converse comparison theorem for BSDEs and related properties of *g-expectations*, *Electron. Comm. Probab*, **5**.

[8] Z. Chen, (1998) A property of backward stochastic differential equations, *C.R. Acad. Sci. Paris Sér. I Math.* **326**, no 4, 483–488.

[9] Z. Chen and L. Epstein (2002), Ambiguity, Risk and Asset Returns in Continuous Time, *Econometrica*, **70**(4), 1403–1443.

[10] F. Coquet, Y. Hu J. Memin and S. Peng (2001) A general converse comparison theore for backward stochastic differential equations, *C.R.Acad. Sci. Paris*, t. 333, Serie I, 557–581, 2001.

[11] F. Coquet, Y. Hu J. Memin and S. Peng (2002) Filtration–consistent nonlinear expectations and related g–expectations, *Probab. Theory Relat. Fields*, **123**, 1–27.

[12] J. Cvitanić, M.C. Quenez and F. Zapatero, (2000) Incomplete information with recursive preference, *preprint.*

[13] El Karoui, N. and Quenez, Dynamic programming and pricing of contingent claims in an incomplete market. SIAM J. Control Optim.(1995) 33 29–66.

[14] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng and M.-C. Quenez (1997), Reflected Solutions of Backward SDE and Related Obstacle Problems for PDEs, *Ann. Probab.* **25**, no 2, 702–737.

[15] Z. Chen and S. Peng (1998) A Nonlinear Doob-Meyer type Decomposition and its Application. *SUT Journal of Mathematics* (Japan), **34**, No.2, 197–208, 1998.

[16] Z. Chen and S. Peng (2001), Continuous Properties of g-martingales, *Chin. Ann. of Math.* **22B: 1**, 115–128.

[17] Z. Chen and S. Peng (2000), A general down crossing inequality for g-martingales, *Statist. Probab. Lett.* **46**, no. 2, 169–175.

[18] C. Dellacherie, (1972) Capacité et Processus Stochastiques, Springer.

[19] C. Dellacherie and P.A. Meyer, (1975) Probabilité et Potentiel, I–IV, Hermann, Paris.

[20] C. Dellacherie and P.A. Meyer, (1980) Probabilité et Potentiel, I–IV, Hermann, Paris.

[21] C. Dellacherie (2002) Nonlinear Dirichlet problem and nonlinear integration, preprint.

[22] D. Duffie and L. Epstein (1992), Stochastic differential utility, *Econometrica* **60**, no 2, 353–394.

[23] N. El Karoui, and M.-C.Quenez (1995), Dynamic Programming and Pricing of Contingent Claims in Incomplete Marke, *SIAM J.of Control and Optimization*, 33, n.1.

[24] N. El Karoui, N., S. Peng and M.-C. Quenez, (1997) Backward stochastic differential equation in finance, *Math. Finance* **7**, no 1, 1–71.

[25] W.H. Fleming and H.M. Soner (1992) *Controlled Markov Processes and Viscosity Solutions,* Springer–Verlag, New York.

[26] Föllmer H. and Alexander Schied, Convex measures of risk and trading constraints, Finance Stoch. 6 (2002), no. 4, 429–447.

[27] Frittelli, M. (2000) Representing sublinear risk measures and Pricing rules, Working paper no. 10, Universita di Milano Bicocca, Italy.

[28] Frittelli, M. and Biaginio, S. On the super replication price of unbounded claims, The Annals of Applied Probability (2004), **14**(4), 1970–1991.

[29] N. Ikeda and S. Watanabe, (1981) *Stochastic Differential Equations and Diffusion Processes,* North–Holland, Amsterdam.

[30] Jiang, L. (2004) Some results on the uniqueness of generators of backward stochastic differential equations, C. R. Acad. Sci. Paris, Ser. I 338 575–580.

[31] Jiang, L. (2004) A converse comparison theorem for g-expectations, Acta Mathematicae Appliticae Sinica, English Series, 20(4), 701˜706.

[32] I. Karatzas and S. E. Shreve, (1988) *Brownian Motion and Stochastic Calculus,* Springer–Verlag, New York.

[33] E. Pardoux and S. Peng (1990), Adapted solution of a backward stochastic differential equation, *Systems and Control Letters* **14**, no 1, 55-61.

[34] S. Peng (1992), A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation, *Stochastic stoch. reports* **38**, no 2, 119–134.

[35] S. Peng (1997), BSDE and Stochastic Optimizations, in *Topics in Stochastic Analysis,* J. Yan, S. Peng, S. Fang and L.M. Wu, Ch.2, (Chinese vers.), Science Publication.

[36] S. Peng (1997), BSDE and related g-expectation, *in Pitman Research Notes in Mathematics Series, no.364, "Backward Stochastic Differential Equation", Ed. by N. El Karoui & L. Mazliak,*, 141–159.

[37] S. Peng, Yang F. (1999) *Duplicating and Pricing Contingent Claims in Incomplete Markets,* Pacific Economic Review, **4**(3) 237–260.

[38] S.Peng (1999), Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type, *Prob. Theory Rel. Fields* **113**, no 4, 473-499.

[39] S. Peng (2002), Nonlinear expectations and nonlinear Markov chains, in *The proceedings of the 3rd Colloquium on "Backward Stochastic Differential Equations and Applications"* , Weihai, 2002.

[40] S.Peng, (2003) The mechanism of evaluating risky values and nonlinear expectations, *Preprint.*

[41] S.Peng, (2003) Nonlinear Expectations, Nonlinear Evaluations and Risk Measures, Lectures in CIME-EMS Summer School, 2003, Bressanone.

[42] Dynamical evaluations, C. R. Acad. Sci. Paris, Ser. I 339 585–589, 2004.

[43] S. Peng, Dynamically Consistent Nonlinear Evaluations and Expectations, arXiv: math.PR/0501415 v1 24 Jan 2005.

[44] Rogers, L.C.G. & Williams, D., *Diffusions, Markov processes and martingales*, Cambridge University Press, 2000.

[45] Rosazza Gianin E. (2003) Some examples of risk measures via g–expectations, preprint.

[46] Yong, J. and Zhou, X. *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer, 1999.